Assessment and Item Specifications for the 2026 NAEP Mathematics Assessment

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CHAPTER 1

OVERVIEW

What Is This Assessment and Item Specifications Document?

This document is a companion to the *Mathematics Framework for the 2026 National Assessment of Educational Progress* (NAEP). The 2026 NAEP Mathematics Framework informs NAEP assessment development, describing the subject matter to be assessed and the questions to be asked, as well as the assessment's design and administration. This Assessment and Item Specifications document extends the Framework, providing greater detail about development of the items and conditions for the 2026 NAEP Mathematics Assessment. The Framework and these accompanying assessment and item specifications are for the National Center for Education Statistics (NCES) and its contractors, critical NAEP partners, who will use both documents to develop the 2026 NAEP Mathematics Assessment.

Background on NAEP

The National Assessment of Educational Progress (NAEP) has measured student achievement nationally since 1973, and state-by-state since the early 1990s, providing the nation with a snapshot of what students in this country know and can do in mathematics. Starting in 2002, urban school districts that meet certain selection criteria could volunteer to participate in the Trial Urban District NAEP Assessment.

There are two distinct components to the NAEP Mathematics Assessment, which differ in purpose. The NAEP Long-Term Trend assessment has measured trends in achievement among 9-, 13-, and 17-year-old students nationally since 1973, and the assessment's content has been essentially unchanged ever since. The second assessment, referred to as "main NAEP," is adjusted over time to reflect shifts in research, policy, and practice. The content and format of the main NAEP Mathematics Assessment are the focus of the Framework.

The main NAEP Mathematics Assessment is administered at the national, state, and selected urban district levels every two years, by Congressional mandate. In mathematics, NAEP results are reported on student achievement in grades 4, 8, and 12 at the national level, and for grades 4 and 8 at the state level and for large urban districts that volunteer to participate.

Taken together, the NAEP assessments provide a rich and broad picture of patterns in U.S. student mathematics achievement. National and state level results are reported in terms of scale scores, achievement levels, and percentiles. These reports provide comprehensive information about what U.S. students know and can do in mathematics. In addition, NAEP provides comparative subgroup data according to gender, race/ethnicity, socioeconomic status, and geographic region; describes trends in performance over time; and reports on relationships between student achievement and certain contextual variables.

The main NAEP assessment is administered to a nationally representative sample of students and reports on student achievement in the aggregate. The assessment is not designed to measure the performance of any individual student or school. To obtain reliable estimates across the population that is assessed, a large pool of assessment items is developed. Subsets of items are administered to each student in the sample. Student results on the main NAEP assessments are reported for three achievement levels established and defined by the National Assessment Governing Board (Governing Board), which oversees NAEP:

- *NAEP Basic* denotes partial mastery of prerequisite knowledge and skills that are fundamental for performance at the *NAEP Proficient* level.
- *NAEP Proficient* represents solid academic performance for each NAEP assessment. Students reaching this level have demonstrated competency over challenging subject matter, including subject-matter knowledge, application of such knowledge to real-world situations, and analytical skills appropriate to the subject matter.
- *NAEP Advanced* signifies superior performance beyond *NAEP Proficient*.

These policy definitions can be found in the Governing Board's *Developing Student Achievement Levels for the National Assessment of Educational Progress Policy Statement* (2018a). Descriptions and examples of student performance at these levels of achievement at grades 4, 8, and 12 for the 2026 NAEP Mathematics Framework are provided in Appendices A and B, respectively. Chapter 5 includes further discussion of the achievement levels.

This document describes specifications for an assessment framework, not a curriculum framework. The 2026 NAEP Mathematics Framework lays out the basic design of the assessment by describing the mathematics content and mathematical practices that should be assessed and the types of questions that should be included. The specifications in this document extend and illustrate these ideas. The Framework also describes how various assessment design factors should be balanced across the assessment. In broad terms, the Framework attempts to answer the question: What mathematics knowledge, skills, and practices are to be assessed on NAEP at grades 4, 8, and 12? The Framework does not cover all relevant content for each grade level; some concepts, practices, and activities in school mathematics are not suitable to be assessed on NAEP, although they may well be important components of a school curriculum. For example, the practice of extended investigation would not be possible in the NAEP assessment, although it would be quite reasonable for teachers to have multi-day investigations of some important mathematical ideas. This document also does not attempt to answer the question: How should mathematics be taught?

The Visioning and Development Process

The process for updating the mathematics assessment framework started with a review of existing frameworks by experts in mathematics education research, policy, and practice, representing key stakeholder groups. This process—which is described in the Governing Board's *Framework Development Policy Statement* (2018b) and elaborated in the 2026 NAEP Mathematics Framework—involved visioning for the update, and then development. For more on the process, see Appendix C. Complementary to the work by the Visioning and Development Panels, a Technical Advisory Committee (TAC) of eight recognized measurement experts advised the panels about technical issues. The TAC made recommendations concerning content and cognitive dimensions in the framework, as well as item and assessment design, and provided feedback on drafts of this specifications document as it was developed.

Overview of Assessment Design and Item Specifications

The Assessment and Item Specifications that guided the development and implementation of the NAEP Mathematics Assessments administered since 2009 were established more than 10 years ago, and significant updates to the Framework have been made for the 2026 assessment. These updates include revisions to the mathematics content objectives, descriptions of new NAEP Mathematical Practices, attention to the evolving role of technology in students' in-school and out-of-school experiences, and consideration of new item formats. These changes required a parallel update to the Assessment and Item Specifications.

The proposed design for the 2026 assessment aims to provide a fair and valid measure of how well all students have achieved the depth and breadth of the mathematics content and practice articulated by the Framework. To do this, the design:

- incorporates a mix of traditional and innovative item types that reflect recent research on the science of learning, to capture both the process and outcomes of student learning, and emphasizes authentic applications of mathematics knowledge and skill;
- capitalizes on the use of technology to assure accessibility, promote engagement for all students, and explore new options for task design and scoring, including the use of multimedia;
- encourages continuing prototyping and research to capitalize on the capacities of current and emerging technology to assess students at deeper levels, while still ensuring validity and fairness of scores; and
- recognizes the potential of technology and new task designs while also acknowledging limitations and potential negative unintended consequences. The design plan is a careful balance to promote more valid assessment of mathematics content and practices without compromising fairness or reliability (e.g., fairness for students who have less access to technology, scenarios that avoid construct-irrelevant barriers of language, and innovative task types that reduce the number of items).

Text and sample items that support and help to clarify the description of the assessment design in the Framework have been included in this Assessment and Item Specifications document. Illustrations include both examples and nonexamples, to assist in the development and implementation of updates for the NAEP Mathematics Assessment.

Introduction to the Assessment and Item Specifications

This Assessment and Item Specifications document includes five chapters and several appendices. Throughout this document, figures have been included to illustrate particular points of emphasis from the Framework. Exhibits that have been carried from the Framework into the Assessment and Item Specifications are labeled as "exhibits" and have the same numbering as in the Framework. Figures that are not included in the Framework are labeled as "illustrations." Illustrations in this document include nonexamples—anti-exemplar items—to support item writers in avoiding items that "function as simpler item types, incorporate superficial complexity that does not improve fidelity to the construct, introduces construct-irrelevant variance, or any combination of the three" (Martineau, Dadey, & Marion, 2018, p. 1). In this document, illustrations are numbered consecutively and separately from exhibits.

Chapter 2 describes the content areas: Number Properties and Operations (including computation and understanding of number concepts); Measurement (including use of instruments and concepts of area and volume); Geometry (including spatial reasoning and applying geometric properties); Data Analysis, Statistics, and Probability (including graphical displays and statistical measures); and Algebra (including representations and relationships). Each content area is broken into subtopics (e.g., for Number Properties and Operations, these are number sense, estimation, number operations, ratios and proportional reasoning, and properties of number and operations) identifying what should be measured on NAEP at grades 4, 8, and 12. Further specifications have been added to some content areas and most objectives, to clarify the measurement intent for item writers.

Chapter 3 describes the NAEP Mathematical Practices that play a role in measuring student knowledge and skills in mathematics. These are Representing, Abstracting and Generalizing, Justifying and Proving, Mathematical Modeling, and Collaborative Mathematics. The chapter argues that content and practices are interwoven and interdependent: one cannot demonstrate mathematics achievement without knowing content and being able to think mathematically. Chapter 3 also offers example items across grades 4, 8, and 12 that illustrate how NAEP Mathematical Practices can be assessed with particular content. Illustrations in this chapter include examples and nonexamples—anti-exemplar items—to support item writers in avoiding potential barriers to NAEP Mathematical Practice alignment.

Chapter 4 focuses on issues of technology and accessibility, assessment design, and item format. The chapter argues for the need to ground the NAEP Mathematics Assessment in tasks in familiar contexts to foster student engagement. By expanding item types and thoughtfully using technology, the NAEP Mathematics Assessment can provide greater access to all students, diversify the ways in which student achievement can be recognized and measured, and more robustly assess both what students know and what they can do. This will involve expanding the assessment to include scenario-based tasks (which involve clusters of related items within one task) along with continued use of existing discrete NAEP items that capture student understanding of content and mathematical practices. As the technology of assessment evolves, alternative formats might also be considered. Illustrations in this chapter include examples and nonexamples, to clarify less familiar item types and best practices in item development.

Chapter 5 addresses how NAEP results are reported. The chapter describes the three NAEP achievement levels and the development of the mathematics achievement level descriptions (see Appendix A). The chapter builds on an expansive conception of "opportunity to learn" as called for by the Visioning Panel Guidelines (see Appendix C). The chapter also discusses how research on student diversity and schooling informs mathematics-specific contextual variables.

Opportunity to Learn and an Expansive Understanding of Contextual Variables

What students learn is inseparable from the conditions of their learning and broader social aspects of mathematics learning. Hence, interpreting differences in what students can do on NAEP requires an understanding of the range of factors that affect student learning. In particular, the Framework articulates an expansive conception of opportunities to learn, informed by educational research on students and their in- and out-of-school learning and experiences, as well as research on the variations in human, material, and social resources that shape what students have an opportunity to learn about mathematics in the U.S. (e.g., Cohen, Raudenbush, & Ball, 2003; Tatto et al., 2012).

Opportunity to learn is generally understood to refer to inputs and processes that shape student achievement, including the school conditions, curriculum, instruction, and resources to which students have access. When opportunity to learn was first used as a concept, Carroll (1963, 1989) emphasized the time allowed for learning. For the past 50 years, the concept of opportunity to learn has continued to evolve, as have efforts to measure in-school opportunities to learn, with the majority of scholars focusing on the classroom as the unit of analysis and instruction as central. Research, for example, has documented the negative effects on achievement of policies and practices that are often found in schools serving children who live in poverty or have special needs, including an inadequate supply of mathematics teachers with strong knowledge and skills, a tendency to offer few advanced mathematics courses, and a common practice of tracking these students disproportionately into low-level courses that restrict their learning opportunities (e.g., Husén, 1967; Tan & Kastberg, 2017), all of which can be understood as instructional resources that shape what students learn.

Important to note is the sociopolitical turn that has taken place in research on school mathematics (Gutiérrez, 2013), which positions mathematics as a “dynamic, political, historical, relational, and cultural subject” (TODOS & NCSM, 2016, p. 3) in which identity and power both play central roles. This turn has led scholars and educators to explore how school mathematics marginalizes and alienates students who do not see connections to their own lives and experiences. It raises questions about how school mathematics might be reformed to engage all students and their communities. This includes students with disabilities who are often relegated to classrooms where learning differences are conceptualized as a deficit rather than a potential strength and where the focus is on procedural approaches rather than leveraging students’ own particular strategies to engage in mathematical reasoning and sense making (e.g., Lambert, Tan, Hunt, & Candella, 2018).

Another noteworthy development in mathematics education research is acknowledgment that students themselves are a resource in learning, including their interests, abilities, and in- and out-of-school experiences. Research, for example, suggests that students’ experiences out-of-school can be directly relevant to the ways they think mathematically and use mathematics (e.g., Martin, 2000; Nasir & Hand, 2008). Some scholars refer to this as students’ “funds of knowledge,” defined as the skills, knowledge, habits of mind, practices, and experiences acquired through historical and cultural interactions of an individual in their community, family life, and culture through everyday living as well as in school (e.g., Aguirre et al., 2013; Civil, 2016; de Freitas & Sinclair, 2016; González, Moll, & Amanti, 2005; Moll, Amanti, Neff, & González, 1992). Students’ funds of knowledge include what has often been referred to as students’ prior knowledge, but expands that idea to include cultural, linguistic, and social traditions that characterize students’ lives out of school. While these funds of knowledge might differ from those of the teacher or the traditional curriculum, the broad experiences of students can be used to make powerful connections that enable learning and can be understood as an additional resource in instruction and assessment. Therefore, the Framework’s conception of opportunity to learn includes students’ experiences, out-of-school learning, and funds of knowledge as an instructional resource.

Relevant opportunity to learn indicators have been clustered in various ways (e.g., Abedi & Herman, 2010; Elliott & Bartlett, 2016; Herman, Klein, & Abedi, 2000; Husén, 1967; Schmidt, Burroughs, Zoido, & Houang, 2015; Wang, 1998). These can be grouped into five strands: time, content and practices, instructional strategies, teacher factors, and instruction-relevant resources. Examples of indicators that have been used in research are provided in Exhibit 1.1.

To support audiences in interpreting NAEP results, information about contextual variables is collected through student, teacher, and administrator surveys. The Framework development process drew broadly on the literature to create an ambitious conception of opportunity to learn as the basis for recommendations about mathematics-specific contextual variables on NAEP surveys. As is the case with mathematics content, it is neither possible nor appropriate to measure all potentially relevant contextual variables on NAEP. For example, questions that ask students about their home or out-of-school experiences can be experienced as intrusive. Priorities for the selection of mathematics-relevant variables are described in Chapter 5.

Exhibit 1.1. Opportunity to Learn Strands

OTL Strand	Example Indicators
Time	time scheduled for instruction proportion of allocated time used for instruction time students are engaged in learning time students are experiencing a high success rate of learning
Content and Practices	content and practices exposure content and practices emphasis content and practices coverage
Instructional Strategies	instructional approaches (e.g., strategies that facilitate student thinking and understanding, instruction that promotes student engagement) classroom climate instructional group size
Teacher Factors	teacher preparation and professional development teacher knowledge, including mathematical knowledge for teaching teaching experience teacher attitudes about themselves, students, learning, and mathematics
Instruction-Relevant Resources	material resources (e.g., textbooks, manipulatives) school policies (e.g., tracking) school community and climate; school and instructional leadership students' experiences, out-of-school learning, and funds of knowledge student access to technological tools

Major Changes in the 2026 NAEP Mathematics Assessment and Item Specifications

This Assessment and Item Specifications document reflects several major changes, both those made to the Framework and those made to support item development. The changes are summarized in the following sections and elaborated in Exhibit 1.2.

Mathematics Content

Chapter 2 presents an updated set of content objectives for the 2026 NAEP Mathematics Assessment at grades 4, 8, and 12. The updates reflect the last decade of changes in state standards for mathematics curriculum, instruction, and assessment. State standards shape what students have had an opportunity to learn by the time they take a NAEP assessment. To ensure the updates reflect current state-level emphases for mathematics content, the Framework incorporates findings from several reports that compared NAEP and state standards (e.g., Achieve, 2016; Johnston, Stephens, & Ratway, 2018), as well as reports on the mathematics content taught in leading countries around the world (e.g., as assessed in the Trends in International Mathematics and Science Study [TIMSS] [NCES, 2019] and the Programme for International Student Assessment [PISA] [OECD, 2019]). Because the Framework has been written for an assessment in 2026 and beyond, it is also informed by national policy that foreshadows likely changes in state policy (e.g., Bargagliotti et al., 2020; Garfunkel & Montgomery, 2019).

Mathematical Literacy

In every state, all high school graduates are required to study mathematics whether or not their future plans involve college or a field in which high school mathematics is heavily involved. The purpose of this universal practice is to ensure that the U.S. citizenry is mathematically literate. Recent policy developments have included attention to mathematical literacy, for example, in mathematical modeling of real-world problems and interpreting reports of data.

Mathematical literacy is the ability to apply mathematical concepts to everyday situations. It has been recognized worldwide as important. In 2015, the PISA assessments, given to 15-year-olds every three years, were conducted in 70 countries, more countries than any other mathematics assessment (OECD, 2018). The PISA assessments emphasize mathematical literacy and define it as the application of numerical, spatial, or symbolic mathematical information to situations in a person’s life as a consumer, employee, or citizen. The definition for the Framework is based on the PISA definition, given its extensive, worldwide use and given the availability of assessment items that have been created following that definition:

> *Mathematical literacy is the application of numerical, spatial, or symbolic mathematical information to situations in a person’s life as a community member, citizen, worker, or consumer.*

A large body of experiences can be viewed as requiring mathematical literacy, including: fluency in the broad range of mathematics of personal finances; understanding statistical information and displays found in print and visual media; and household tasks such as cooking, cleaning, and furnishing that require a variety of measurements. For example, mathematical literacy affects how one critically evaluates reports on environmental issues, estimates how many bricks are needed to build a walkway, or compares interest rates for a loan. Mathematical literacy is part of the everyday experiences that occur in community, civic, professional, and personal contexts of adults in the United States, regardless of career.

At grades 4 and 8, instances of mathematical literacy are found in the standard content taught in schools, have been in previous NAEP frameworks, and remain in the objectives enumerated here. At grade 12, historically, instances of mathematical literacy have been given less attention.

In the 2026 NAEP Mathematics Framework, throughout grade 12, objectives that provide opportunities for assessment of mathematical literacy are identified by the number/hashtag sign (#). See Chapter 2 for more on the issue of mathematical literacy.

NAEP Mathematical Practices

Since the late 1980s, there have been ongoing efforts to more clearly specify mathematical processes like "higher-order thinking" or "mathematical reasoning." Current conceptions of mathematical knowledge and skill have shifted to specify mathematical practices and processes. At the turn of the 21st century, in *Adding It Up*, the National Research Council (NRC, 2001) enumerated five strands of mathematical proficiency, including:

- *conceptual understanding*: comprehension of mathematical concepts, operations, and relations;
- *procedural fluency*: skill in carrying out procedures flexibly, accurately, efficiently, and appropriately;
- *strategic competence*: ability to formulate, represent, and solve mathematical problems;
- *adaptive reasoning*: capacity for logical thought, reflection, explanation, and justification; and
- *productive disposition*: habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.

For decades, the National Council of Teachers of Mathematics (NCTM) has discussed five mathematical processes standards: problem solving, reasoning and proof, communication, connections, and representation (NCTM, 2000). Processes like these have been central to NAEP frameworks for the last 20 years and state standards have reiterated the important role of practices. The language of "practice" has become increasingly popular, establishing a foothold through various state standards, as well as in discussions of teaching with and through practices (NCTM, 2014). The Framework provides the following definition:

> *NAEP Mathematical Practices are the routines, norms, and processes needed to do the work of mathematics.*

Based on the current state of the field, the Framework identifies five NAEP Mathematical Practices for the NAEP Mathematics Assessment:

NAEP Mathematical Practice 1: Representing
NAEP Mathematical Practice 2: Abstracting and Generalizing
NAEP Mathematical Practice 3: Justifying and Proving
NAEP Mathematical Practice 4: Mathematical Modeling
NAEP Mathematical Practice 5: Collaborative Mathematics

These mathematical practices are described in depth in Chapter 3. Note that these mathematical practices are not instructional practices used by teachers. They are the actions necessary to do mathematics. This list of NAEP Mathematical Practices also does not endorse one particular view of mathematical practices (an issue further discussed in Chapter 3).

Item Formats and Technology in Assessment

A fourth major change involves item formats and the role of technology in assessment. As noted previously and as further explained in Chapter 4, technological innovation is relevant to NAEP because it allows for more authentic assessments and for a broader range of accommodations to meet students' needs.

Since 1992, the NAEP Mathematics Assessment has used two types of items (questions): multiple choice and constructed response. In 2017, the NAEP assessment began to include these item formats in a digital platform as part of the NAEP transition to digitally based assessment. The transition to digital administration provided opportunities to expand the range of formats used for items.

In advancing the expansion of item types and formats, three themes emerged. One theme concerns how research on students' knowledge and experience can be used to design assessments that capture their capacity to do mathematics. This includes the use of interactive, multimedia scenario-based tasks to assess what students know and can do. Scenario-based tasks currently exist in other NAEP assessments, including NAEP Science and NAEP Technology and Engineering Literacy.

By expanding item formats, to include scenario-based tasks (and new item formats that emerge in the future) and to thoughtfully use technology, the aim is to provide greater access to all students, as well as to diversify the ways in which student achievement can be recognized and measured. Note that technological innovation is not just limited to enhancing assessment accommodations. Technology is a part of every student's life and learning, and mathematical thinking can be enhanced by its judicious use.

A second theme concerns the use of technology to enable assessment of the NAEP Mathematical Practices, including an expanded range of response types leveraging object-based and discourse responses within a scenario-based task. Less often noted but equally important is a third theme concerning the intended or unintended negative consequences of technology, which include inequitable access to technologies. That is, while technology may have the potential to increase access and opportunities to demonstrate learning, students unfamiliar with technologies used in the assessment could be at a disadvantage. With the introduction of scenario-based tasks it is critical to ensure that students have ample time to understand how to engage with assessment items along with opportunities to experience the task type.

Changes from the 2009–2017 Assessment and Item Specifications

Exhibit 1.2 compares the Framework and Assessment and Item Specifications for the 2026 NAEP Mathematics Assessment and those used for the 2009–2017 NAEP Mathematics Assessments. The focus here is on major changes. Many of the points summarized in Exhibit 1.2 are expanded in Chapters 2, 3, and 4. Justifications for these changes are briefly described in Exhibit 1.2, with more details in the relevant chapters.

Exhibit 1.2. Summary of Changes in the 2026 NAEP Mathematics Framework and Assessment and Item Specifications

Topic	Change	Rationale
Mathematics Content	Many objectives were edited to increase clarity and specificity.	Objectives and balance of topics were updated to reflect shifts in expectations evident from reviews of state and national standards, policy documents from leading professional organizations, and expectations for mathematical literacy on U.S. and international assessments. For more details on changes, see Chapter 2.
	Additional clarifications and limitations were included with the content objectives to further guide item development.	Suggestions were included to reflect content descriptions from the previous Assessment and Item Specifications (2009), 2026 Framework authors, state standards and supporting documents, and public-facing information from current state and national assessments.
	The objectives in the mathematical reasoning subtopics have been removed. This subtopic was introduced in 2009 for Number Properties and Operations; Geometry; Data Analysis, Statistics, and Probability; and Algebra.	With the introduction of the NAEP Mathematical Practices (see Chapter 3), mathematical reasoning was no longer needed as a subtopic. To preserve attention to the content that was uniquely present in some of the mathematical reasoning objectives, objectives in other subtopics were revised. For more details on changes, see Chapter 2.
	Distribution of items for grade 12 remains the same. The proportion of Data Analysis, Statistics, and Probability items has increased for grade 8 and decreased for grade 4. Concurrently, the proportion of items in Measurement in grade 8 decreased and the proportion in Number Properties and Operations in grade 4 increased.	Adjustments to the proportion of items on the assessment in Data Analysis, Statistics, and Probability at grades 4 and 8 reflect changes in opportunity to learn common across state standards. The distribution of attention to content topics in state standards informed the related decisions to increase the proportion of items at grade 4 in Number Properties and Operations and decrease the proportion in Measurement at grade 8. For more details on changes, see Chapter 2.

Exhibit 1.2. Summary of Changes (continued)

Topic	Change	Rationale
Mathematics Content (continued)	Illustrations containing items and associated text providing clarification for specific text from the Framework were included.	Illustrations containing example and nonexample items, as well as discussions of these items, were included to assist item writers in developing a richer understanding of what was (and was not) intended by the Framework.
Mathematical Complexity (2017 Framework)	This was a chapter that defined mathematical complexity as "the demands on thinking that an item expects" (Governing Board, 2017a, p. 37). The chapter was removed.	From 2009 to 2017, "mathematical complexity" aimed to address the process dimension, the "doing" of knowing and doing mathematics. It was a mixing of cognitive demands (e.g., on working memory, reading comprehension, and attention) and the challenges inherent in developing mathematical understanding. However, it was not supportive of score interpretation. Many decades of research and development have shown that assessing students' knowledge and use of mathematics is more nuanced than was accounted for in the "mathematical complexity" approach used in previous frameworks.
NAEP Mathematical Practices (NEW)	A new chapter, Chapter 3 – NAEP Mathematical Practices, has been added describing and illustrating the assessment of five mathematical practices through which students engage in knowing and doing mathematics.	Since the 1990s, the field of mathematics education has seen increasing focus on mathematical processes and the interacting social and mental activities of knowing and doing mathematics. This chapter reflects the field's attention to mathematical activity by describing five NAEP Mathematical Practices. These are assessable aspects of activity at work across mathematics content when students do mathematics.

Exhibit 1.2. Summary of Changes (continued)

Topic	Change	Rationale
NAEP Mathematical Practices (continued)	A distribution of items for each mathematical practice was developed.	Most NAEP Mathematics Assessment items will feature at least one of the five NAEP Mathematical Practices (55 to 85 percent). This range allows flexibility in assessment and item development across grades 4, 8, and 12 while also ensuring that the majority of the assessment is designed to capture information on student knowledge while engaging in mathematical practices. The balance of items (15 to 45 percent) will assess knowledge of content without calling on a particular mathematical practice (e.g., procedural or computational skill).
	Items illustrative of a NAEP Mathematical Practice or serving as nonexamples of a practice were introduced within the text for each practice.	These items were included to provide additional support for item writer conceptualization of the NAEP Mathematical Practices.
Item Formats and Assessment Design	Two chapters in the previous framework (Item Formats and Design of Test and Items) were merged into a single chapter, Chapter 4 – Overview of the Assessment Design, and updated.	The combination of chapters on assessment and item design allowed addressing interrelationships among: (1) the new digital format of NAEP administration, and (2) developments in technology for assessment, including scenario-based tasks.
	A new format, scenario-based task, was introduced.	With the addition of scenario-based tasks, the NAEP Mathematics Assessment continues to provide greater access to all students, diversifies the ways in which student achievement can be recognized and measured, and more robustly assesses both what students know and what they can do.

Exhibit 1.2. Summary of Changes (continued)

Topic	Change	Rationale
Calculator Policy	Continuing the policy established for the 2017 digital administration of NAEP, students will have access to a calculator emulator in blocks of items designated as "calculator blocks": four-function for grade 4, scientific for grade 8. The one change in 2026 and beyond will be that the grade 12 calculator will include a graphing emulator.	High school students typically use graphing calculators or online emulators and not scientific calculators (Crowe & Ma, 2010).
Item Types	Chapter 4 includes updates to reflect current and future digital platform use and the new format option of scenario-based tasks.	To better assess the diversity of ways of doing mathematics, technology available now and in the near future allows scenario-based tasks. Scenario-based item collections can be used to assess aspects of mathematical activity that have been difficult (if not impossible) to assess in the past. Building on the work in the last five years to use scenario-based tasks in NAEP Science and NAEP Technology and Engineering Literacy assessments, Chapter 4 details the ways scenario-based and traditional items can be combined to assess achievement in mathematics content and NAEP Mathematical Practices.
	Items illustrative of an item type or best practice in development of items for the NAEP Mathematics Assessment were introduced. Illustrations serving as nonexamples of best practices in development of items for the NAEP Mathematics Assessment were also included.	Illustrations containing example and nonexample items, and discussions of these items, were included to provide additional support for application of best practices in item writing for the NAEP Mathematics Assessment and actualization of potential NAEP mathematics item types.

Exhibit 1.2. Summary of Changes (continued)

Topic	Change	Rationale
Tools and Manipulatives	Students will continue to have the tools and manipulatives used in the digital administration of the 2017 NAEP Mathematics Assessment. Chapter 4 also explores the potential of behind-the-scenes technology to capture and use process data for assessment; these are data generated by students as they work with the assessment.	The existing digital system tools and mathematics-specific tools have proven worthwhile since the 2017 administration. Additionally, in acknowledgment of the continuing evolution and use of technology in mathematics, Chapter 4 includes examples of other tools (e.g., simulations, dynamic geometry software, and "smart" physical objects) that may be common in 2026 and beyond.

Aligning with the Framework and the Assessment and Item Specifications

The assessment should be developed so that it is aligned with the guidelines defined by the intersection of content objectives and NAEP Mathematical Practices, as set forth in the Framework and in these Assessment and Item Specifications. More specifically:

- The content of the assessment should be matched with the content of the Framework and the Assessment and Item Specifications. The assessment as a whole should reflect the breadth of knowledge covered by content objectives, clarifications, and limitations in the Framework and the Assessment and Item Specifications. The content of the assessment should not go beyond the content boundaries as defined in these documents. The assessment should represent the balance of mathematics content at each grade as described in Chapter 4 of the Framework and the Assessment and Item Specifications.
- The mathematical practices reflected in items on the assessment should be matched to the NAEP Mathematical Practices in the Framework and the Assessment and Item Specifications. The assessment should represent the balance of NAEP Mathematical Practices at each grade as described in Chapter 4 of the Framework and the Assessment and Item Specifications.
- While it is not possible to cover all possible combinations of content objectives and practices for each achievement level on one assessment, appropriate alignment between the assessment and the Framework and Assessment and Item Specifications at each grade should be maintained in the item pools. The assessment should be built so that the constructs represented by the objectives for each content area are adequately represented. The breadth and relative emphasis of mathematics knowledge covered in each content area, as presented in the Framework and the Assessment and Item Specifications, should be represented on the assessment as a whole. The developer should avoid under- or overemphasizing particular content objectives, NAEP Mathematical Practices, or achievement expectations, the goal being to ensure broad coverage in any given year's item pool and coverage of all content objectives over time.

- The assessment should represent the balance of response types specified in Chapter 4 of the Framework and the Assessment and Item Specifications, and should give appropriate emphasis to the testing time allocated for scenario-based tasks.
- The assessment should report and interpret scores based on the Framework, the Assessment and Item Specifications, and the NAEP Achievement Level Descriptions (ALDs). That is, the assessment should be developed so that scores will reflect both the guidelines in the Framework and Assessment and Item Specifications and the range of performances illustrated in the NAEP Mathematics ALDs.
- The assessment design should match the characteristics of the targeted assessment population. That is, the assessment should give all students tested a reasonable opportunity to demonstrate their knowledge and skills in the content areas and NAEP Mathematical Practices covered by the Framework and the Assessment and Item Specifications.

The Nation's Report Card (Governing Board, n.d.) is a valuable online resource for learning more about NAEP. This site contains reports describing results of recent assessments, as well as a searchable tool for viewing released items. The items can be searched by many different criteria, such as grade level and content area. Information about the items includes student performance data and any applicable scoring rubrics. NAEP released items that are used as examples and nonexamples in this document are marked with the designation that matches the item name or identified by the question ID from the NAEP Questions Tool website (NCES, n.d.).

MATHEMATICS CONTENT

The NAEP Mathematics Assessment measures what mathematics students know and are able to do, which involves understanding of particular mathematical ideas (content) and of how to use those ideas in mathematical activity (practices). The content of mathematics can be described by nouns: numbers, data, variables, functions, graphs, geometric figures of various kinds, and the like. In contrast, mathematical practices can be described by verbs: recognize, generalize, deduce, justify, and other processes of mathematical reasoning; represent, use, symbolize, and other actions involved in applying mathematics; describe, explain, model, and other activities inherent in mathematics being a discipline that is socially constructed by, and communicated among, individuals and societies.

This chapter focuses on the mathematics content objectives; Chapter 3 focuses on the NAEP Mathematical Practices. Mathematical proficiency involves knowing both.

Content Areas

NAEP has regularly gathered data on students' understanding of five broad areas of mathematics content:

- **Number Properties and Operations** (including computation and understanding of number concepts)
- **Measurement** (including use of instruments, application of processes, and concepts of area and volume)
- **Geometry** (including spatial reasoning and applying geometric properties)
- **Data Analysis, Statistics, and Probability** (including graphical displays)
- **Algebra** (including expressions, equations, representations, and relationships)

Classification of an item into one primary content area is not always clear-cut, but it helps to ensure that the indicated mathematical concepts and skills are assessed in a balanced way.

Certain aspects of mathematics occur in all content areas. For example, there is no single objective for computation. Instead, computation is embedded in many content objectives. In the Framework, computation appears in the Number Properties and Operations objectives, which encompass a wide range of concepts about the numeration system and explicitly include a variety of computational skills, ranging from operations with whole numbers to work with decimals, fractions, percents, and real and complex numbers. Computation is also critical in Measurement and Geometry in determining, for example, the perimeter of a rectangle, estimating the height of a building, or finding the hypotenuse of a right triangle. Data analysis often involves computation in calculating a mean, or other statistics describing a collection of values, or in calculating probabilities. Solving algebraic equations also frequently involves numerical computation.

The objectives describe what is to be assessed on NAEP given operational limitations. As noted in Chapter 1, the NAEP content objectives are not a complete description of mathematics that should be taught at these grade levels.

NAEP Mathematics Assessment Objectives Terminology

Some terms that are broadly used in mathematics education must take on narrower meanings in order to clearly describe measurable mathematics objectives. To support item development aligned with the objectives given in this document, several points bear mention:

- The phrase "solve problems" means to complete tasks where the task contexts may range from the purely mathematical to those that are experientially concrete or real to students.
- When the word "or" is used in an objective, it means that an item may assess one or more of the concepts included, and the full collection of items will include assessment of each listed concept. The table in Illustration 2.1 provides example objectives to further clarify this intention.
- Specific to grade 12 are three distinctions in NAEP content objectives:
    - Some grade 12 objectives are marked with an asterisk (*). This denotes objectives that describe mathematics content beyond what is typically taught in a 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra, with statistics and probability included). These objectives will be selected less often than the others for inclusion on the assessment. For item development, the asterisk applies to the entire objective if it appears immediately after the objective letter, or applies to an immediately following word or phrase if it prefaces a word or phrase within the objective description. For example:
        - In grade 12 Meas – 3.d, the entire objective is indicated: "d) * Interpret and use…"
        - In grade 12 Num – 1.d, only the "logarithms" aspect of the objective carries an asterisk: "d) Represent, interpret, or compare expressions for real numbers, including expressions using exponents and *logarithms."
        - In grade 12 Num – 2.b, of three aspects listed for the objective, only the third, "analyze the effect," carries the asterisk: "b) Identify situations where estimation is appropriate, determine the needed degree of accuracy, and *analyze the effect of the estimation method on the accuracy of results."
    - Some objectives in grade 12 are marked with the number/hashtag sign (#). This designates objectives that most closely reflect opportunities to assess mathematical literacy. However, not all items associated with an objective that has the # sign will assess mathematical literacy.
    - At grade 12, geometry and measurement are combined as one content area. This reflects the fact that the majority of measurement topics suitable for high school students are geometric in nature.
- Although every assessment item will be assigned a primary classification, some items could potentially fall under more than one objective.

As mentioned in Chapter 1, "illustration" is used throughout the Assessment and Item Specifications to indicate exhibits that are not in the Framework. These include examples and nonexamples intended to further clarify particular points of emphasis in the Framework. Each exhibit carried from the Framework into the Assessment and Item Specifications remains labeled as an "exhibit."

Illustration 2.1. Example: Multi-Verb Objectives and the Use of "Or"

Grade Level	Number Properties and Operations Objective	Clarifications
4	3e) Interpret, explain, or justify whole number operations and explain the relationships between them.	The item pool will include items that measure each of the four targets of this objective: (1) interpreting whole number operations, (2) explaining whole number operations, (3) justifying whole number operations, and (4) explaining the relationships between whole number operations.
8	3e) Interpret, explain, or justify rational number operations and explain the relationships between them.	The item pool will include items that measure each of the four targets of this objective: (1) interpreting rational number operations, (2) explaining rational number operations, (3) justifying rational number operations, and (4) explaining the relationships between rational number operations.
12	3e) *Analyze or interpret a proof by mathematical induction of a simple numerical relationship.	The item pool will include items that measure each of the two targets of this objective: (1) analyzing a proof by mathematical induction of a simple numerical relationship, and (2) interpreting a proof by mathematical induction of a simple numerical relationship.

Mathematical Literacy

As noted in Chapter 1, mathematical literacy is related to an individual's capacity to "understand the role that mathematics plays in the world, to make well-founded judgments and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned citizen" (OECD, 2003, p. 3). It includes the ability to formulate and interpret problems, and to use mathematical knowledge and skill in creative ways across a range of situations—complex and simple, routine and unusual. These situations can occur in one's private life (measuring cloth for a project), one's occupational and professional life (using proportions to make sense of a situation), one's social life with friends or family (paying in a restaurant), and in one's life as a citizen (processing information relevant to voting).

Some objectives at grade 12 are identified with the theme of mathematical literacy. If there are everyday applications of the objective to situations in a person's life as a community member, citizen, worker, or consumer, then the number/hashtag sign (#) precedes the objective. For example, for an objective that calls for students to analyze situations, develop mathematical models, or solve problems using a particular form of equation or inequality, mathematical literacy items might be given in real-world contexts such as solving a problem about tax implications of a workplace policy change, or, in the context of community decisions, analyzing or modeling with an inequality the upper bounds for safe levels of lead in water from a local water treatment facility. Other items *not* focused on mathematical literacy might ask the student to solve a problem by graphing the consequences of doubling the value of a variable in a linear relationship.

As another example, a mathematical literacy assessment item might provide information about a seismic magnitude scale (used to measure the intensity of earthquakes), indicate that on the scale a Magnitude 5 earthquake is ten times stronger than a Magnitude 4 earthquake, and ask grade 12 students to make sense of, model, or draw conclusions in a problem situation that uses that information. An alternate assessment item for the same objective that would not be focused on mathematical literacy might ask students to apply and justify the use of logarithms to determine the seismic magnitude measurement in a given situation. The goal of the identification of objectives with # is to support exploration of NAEP reporting on mathematical literacy. See Appendix E for a description of a special study on assessing and reporting on mathematical literacy.

Item Distribution

The distribution of items among the various mathematics content areas is a critical feature of the assessment design because it reflects the relative importance given to each area in the assessment. As has been the case with past NAEP assessments, the categories have different emphases at each grade. Exhibit 2.1 provides the balance of items in the assessment by content area for each grade (4, 8, and 12). The percentages refer to the proportion of items, not the amount of testing time.

For the 2026 NAEP Mathematics Assessment, a greater number of items assessing fraction concepts will be sampled than have been in past administrations. This increase reflects not only the focus on fraction instruction in the early grades, but also the importance of understanding students' early knowledge of and skills with fraction concepts, as they are a predictor of success in high school mathematics courses (Siegler et al., 2012).

Exhibit 2.1. Percentage Distribution of Items by Grade and Content Area

<table>
<tr><th>Content Area</th><th>Grade 4</th><th>Grade 8</th><th>Grade 12</th></tr>
<tr><td>Number Properties and Operations</td><td>45*</td><td>20</td><td>10</td></tr>
<tr><td>Measurement</td><td>20</td><td>10</td><td rowspan="2">30</td></tr>
<tr><td>Geometry</td><td>15</td><td>20</td></tr>
<tr><td>Data Analysis, Statistics, and Probability</td><td>5</td><td>20</td><td>25</td></tr>
<tr><td>Algebra</td><td>15</td><td>30</td><td>35</td></tr>
</table>

*Note: At least one-third of grade 4 Number Properties and Operations items should assess fraction content.

NAEP Mathematics Objectives Organization

Mathematical ideas in different content areas are often interconnected. Organizing the Framework by content areas has the potential for obscuring these connections and leading to fragmentation. However, the intent here is that the objectives and the assessment of those objectives will, in many cases, cross content area boundaries.

To provide clarity and specificity in grade-level objectives, the Framework matrix (Exhibits 2.2, 2.3, 2.4, 2.5, and 2.6) depicts the objectives appropriate for assessment under each subtopic. For example, within the Number Properties and Operations subtopic of Number Sense, specific objectives are listed for assessment at grades 4, 8, and 12. In general, objectives within content areas are different across the grades. Occasionally, the same objective may appear at more than one grade level; this suggests an implicit developmental sequence for that concept or skill. An empty cell in the matrix conveys that an objective is not appropriate or not deemed as important as other areas for assessment at that grade level. Explanations of changes in the mathematics objectives are elaborated in the final section of this chapter.

Objective Alignment and Illustrations

Throughout this Assessment and Item Specifications document, assessment items have been included to illuminate particular text in the Framework. The items used in illustrations come from a variety of sources, including released items from the NAEP Questions Tool (NCES, n.d.), suppliers of state assessments (e.g., SBAC, 2018; PARCC, 2015), and international mathematics assessments, such as TIMSS, PISA, and England's *Key Stage* tests. Sources are named with the description of each item, and a note is included when the item has been modified for the purposes of this document.

At the top of most illustrations is a metadata table with key information about the item used. These metadata are specific to the 2026 NAEP administration and identify five pieces of information (see Illustration 2.2a).

- **Grade Level**: identifies the 2026 grade level
- **Content Area**: identifies the 2026 primary content area. Abbreviations for each content area used throughout this document are included in parentheses.
    - Number Properties and Operations (Num)
    - Measurement (Meas)
    - Geometry (Geom)
    - Data Analysis, Statistics, and Probability (Data)
    - Algebra (Alg)
- **Assessed Practice(s)**: identifies the assessed NAEP Mathematical Practice(s)
- **Objective ID**: identifies the 2026 NAEP content objective alignment
- **Item Format**: identifies the 2026 item format. Abbreviations used for item formats are listed below. See Chapter 4 for a description of each.
    - SR: selected response
        - SR – MC (multiple choice)
        - SR – MS (multiple select)
        - SR – matching
        - SR – zone
        - SR – grid
        - SR – IC (in-line choice)
        - SR – composite
    - SCR: short constructed response
        - SCR – FIB (fill-in-the-blank)
        - SCR – composite
    - ECR: extended constructed response
        - ECR – ET (extended text)
        - ECR – composite

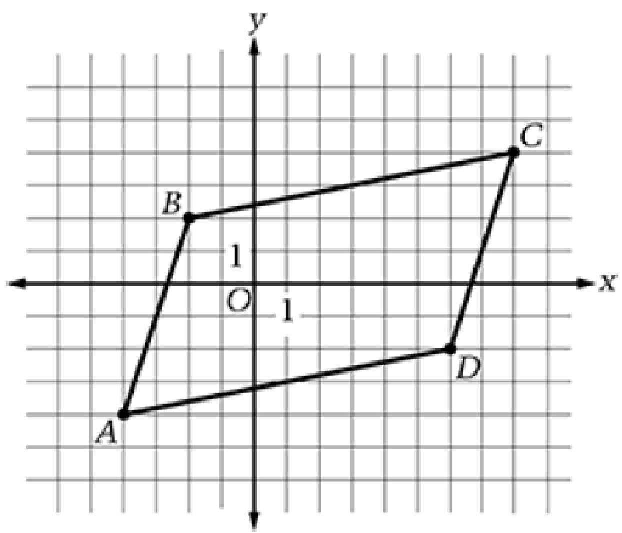
Illustration 2.2a. Example: Item Metadata

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
12	Geometry	Other	Geom – 3.h	SCR

As noted in Chapter 1, for the 2026 assessment, the "Mathematical Reasoning" subtopics were removed. The intent of objectives in the Mathematical Reasoning subtopics was addressed in the 2026 Framework through additions to other subtopics and through the NAEP Mathematical Practices (see Chapter 3 for more on the NAEP Mathematical Practices). Consequently, the Objective ID for a 2026 item may differ from the Objective ID for an older item. For example, for the item whose metadata are shown in Illustrations 2.2a and 2.2b, the 2026 Objective ID is Geom – 3.h. However, when that item was administered on the 2009 NAEP Mathematics Assessment, its Objective ID was Geom – 5.a. because in 2009 the Framework included Mathematical Reasoning as the fifth subtopic.

Illustration 2.2b. Example: Original Objective ID and 2026 Objective ID Differ

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
12	Geometry	Other	Geom – 3.h	SCR



In the figure above, the vertices of $ABCD$ are $A(-4, -4)$, $B(-2, 2)$, $C(8, 4)$, and $D(6, -2)$.

Give a mathematical justification that $ABCD$ is a parallelogram.

Scoring Information	
Key	[Scoring Rubric: for more information on scoring information, see Chapter 4]

The item in this illustration is based on a NAEP item. The original version of this item appeared in the 2009 grade 12 NAEP Mathematics Assessment with NAEP Item ID 2009-12M2 #12 M195001.

Another difference worth noting is the adjusting from grade 4 to grade 8 of some objectives in probability. In grade 4, a review of state and national mathematics standards indicated an absence of student opportunity to learn the content of probability objectives. Therefore, probability items originally developed for grade 4 may now be aligned to objectives that appear at grade 8 in the 2026 Framework. Illustration 2.3 gives an example.

Illustration 2.3. Example: Probability Objective Moved from Grade 4 to Grade 8

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
8	Data Analysis, Statistics, and Probability	Other	Data – 4.e	SCR – composite

Al, Bev, and Carmen are going on a ride at the park. Only 2 people can go on the ride at a time. They can pair up 3 different ways, as shown below.

Al and Bev

Al and Carmen

Bev and Carmen

Derek decides to join the group. How many different ways can the 4 students pair up?

Answer: ___________________

Show your work or explain how you got your answer.

Scoring Information	
Key	[Scoring Rubric: for scoring information, see Illustration 4.17c in Chapter 4]

The item in this illustration is based on a NAEP item. The original version of this item appeared in the 2013 grade 4 NAEP Mathematics Assessment with NAEP Item ID 2013-4M6 #14 M136901.

Similar Objectives Across Multiple Grade Levels

Several concepts included in NAEP objectives span multiple grade levels. In this document, through the language used in the objectives or in additional notes for item development, the content is differentiated at each grade level. For example, Number Properties and Operations objective 1.i at each grade level involves ordering and comparing numbers. These objectives are shown in Illustration 2.4.

Illustration 2.4. Number Properties and Operations Objectives 1.i

Grade 4	Grade 8	Grade 12
i) Order or compare whole numbers, decimals, or fractions using common denominators or benchmarks.	i) Order or compare rational numbers including very large and small integers, and decimals and fractions close to zero.	# i) Order or compare rational or irrational numbers, including very large and very small real numbers.

The objectives are worded similarly. The differences are in the types of numbers being compared. At grade 4, students compare whole numbers, decimals, or fractions; at grade 8, the sets of numbers are expanded to include rational numbers; and at grade 12, irrational numbers are included.

Specifications Added to Content Objective Exhibits

Exhibits for the content objectives from the Framework have been augmented in this document (Exhibits 2.2, 2.3, 2.4, 2.5, and 2.6). The presentation of these specifications includes italicized text, indicator symbols such as caret (^) and plus (+), and some frequently recurring phrases.

Italicized text provides clarifications or limitations to inform item development. All such text is from the 2009 Assessment and Item Specifications document (Governing Board, 2007), except for text that includes a leading symbol.

- A leading caret (^) indicates edited text from the 2009 Assessment and Item Specifications.
- A leading plus (+) indicates text new to the 2026 Assessment and Item Specifications.

Specifications related to the wording of statements in italicized text are described below.

- "Items should" and "Items should not" statements provide constraints and limitations for the assessment of the associated objectives.
- "Emphasis should be on" statements indicate characteristics of a majority of the items in the item pool for the associated objectives.
- Statements that indicate that an item or other object of interest "can" be, do, or contain something indicate allowance for the described action or description. These include "For example" statements that provide examples of ways that objectives might be assessed.
- "Include items that" statements indicate characteristics of at least some of the items in the item pool for the associated objectives.
- "See" statements refer the reader to a specific location in the chapter for additional information.

Many objectives and clarifications indicate that developed items should have a context. At times, the word "context" is modified by an adjective to provide specific information regarding the type of context required.

- "Real-world context" refers to situations that are concrete or that include specific details related to human perception, activities, or relationships with the physical world. These specific details are necessary in order for students to understand or complete the item.
- "Mathematical context" refers to purely mathematical or abstract item settings that are not connected with students' everyday life experiences. In these cases, the mathematics is central to the item; the context may provide a setting for the mathematics but is often thin and does not need to be interpreted to solve the problem.
- "Familiar context" and "meaningful context" may be either a real-world context, a mathematical context, or a combination of the two. In these cases, students have experience with the context, or the context has meaning for the students.

The sources of these suggestions include the previous Assessment and Item Specifications (Governing Board, 2007), 2026 NAEP Mathematics Framework authors, public-facing information from current state and national assessments (e.g., state assessment websites; PARCC, 2015; SBAC, 2018; and SBAC-related *Progressions* documents [Common Core Standards Writing Team, 2019]), mathematical modeling guidelines (Garfunkel & Montgomery, 2019), and preK–12 statistics guidelines (Bargagliotti et al., 2020).

Number Properties and Operations

Numbers (used as counts, measures, ratio comparisons, and scale values) are tools for describing the world quantitatively. It is thus not surprising that *Number* constitutes a major content focus of school mathematics, especially through grade 8. This focus includes facility with different notational forms (as whole numbers, fractions, decimals, percents, powers, and radicals), an understanding of number systems (e.g., integers, rational numbers, real numbers) and their properties, and calculational proficiency with these forms within systems.

Ancient cultures around the world had names for numbers and ways of doing arithmetic. The accessibility and usefulness of arithmetic today is greatly enhanced by the worldwide use of the Hindu-Arabic decimal place value system. In its full development, this remarkable system includes finite and infinite decimals that allow approximating any real number as closely as desired. Decimal notation simplifies arithmetic by means of routine algorithms; it makes size comparisons straightforward and estimation simple.

Numbers are not simply labels for quantities; they form systems with their own internal structure. For instance, at times problems can be more easily solved by considering what numbers add up to a certain value (e.g., 100 – 98 can be thought of as "98 plus what adds up to 100?"). Multiplication is connected to the idea of repeated addition just as division is connected to the idea of repeated subtraction, and the relationship between multiplication and division can be used to simplify computation (e.g., instead of multiplying a number by 25, a number can be multiplied by 100 and then divided by 4, perhaps by halving and halving again). Arithmetic operations (addition, subtraction, multiplication, and division) and the relationships among them help students determine the mathematics that corresponds to basic real-world actions. For example, joining two collections or laying two lengths end-to-end can be described by addition, while comparing two collections can be described by subtraction, and the concept of rate depends on division. Multiplication and division of whole numbers lead to the beginnings of number theory, including concepts of factorization, remainder, and prime number. Another basic structure of real numbers is ordering, as in which is greater or lesser. Attention to the relative size of quantities provides a basis for making sensible estimates.

Number is not an isolated mathematics domain; it is intimately interwoven with other content strands. In their study of measurement, students use numbers to describe continuous quantities such as length, area, volume, weight, and time, and even to describe more complicated derived quantities such as rates of speed, density, inflation, interest, and so on. With numbers, students can count collections of discrete objects or describe fractional parts of data sets, allowing for statistical analysis. As elementary-grade students generalize number relationships and properties they engage in algebraic thinking. In pursuit of graphical depictions of algebraic relationships, students use Cartesian coordinates—ordered pairs of numbers to identify points in a plane and ordered triples of numbers to label points in space. Numbers allow precise communication about anything that can be counted, measured, or located in space.

Comfort in dealing with numbers effectively is called *number sense*. It includes intuition about what numbers mean; understanding the ways to represent numbers symbolically (including facility with converting between different representations); the ability to calculate, either exactly or approximately, and by several methods (e.g., mentally, with paper and pencil, or calculator, as appropriate); and the ability to estimate. Skill in working with proportions (including percents) is another important part of number sense.

Number sense is a major expectation of the NAEP Mathematics Assessment. In grade 4, students are expected to have a solid grasp of whole numbers as represented in the base 10 system and to begin understanding fractions. By grade 8, students should be comfortable with rational numbers, represented either as decimal fractions or as common fractions, and should be able to use them to solve problems involving proportionality, percentages, and rates. At this level, number sense should also begin to coalesce with geometry by extending students' understanding of the number line. This concept is connected with approximation and the use of scientific notation. Grade 8 students should also have some acquaintance with naturally occurring irrational numbers, such as square roots and π (pi). By grade 12, students should be comfortable dealing with all types of real numbers and various representations, for example, as powers. Students in grade 12 should be able to establish the validity of numerical properties using mathematical arguments.

The 2026 Number Properties and Operations objectives are shown in Exhibit 2.2. Included with many of the objectives is italicized text providing clarifications or limitations for use during item development.

Exhibit 2.2. Number Properties and Operations (Num)

Num – 1. Number sense		
Grade 4	**Grade 8**	**Grade 12**
a) Identify place value and actual value of digits in whole numbers, and think flexibly about place value notions (e.g., there are 2 hundreds in 253, there are 25 tens in 253, there are 253 ones in 253). *+Items should limit numbers to whole numbers through* 999,999. *+Emphasis should be on numbers through* 999.	a) Use place value to represent and describe integers and decimals.	
b) Represent numbers using base 10, number line, and other representations. *+Items should limit numbers to whole numbers through* 999. *+Items should involve representations that students can use intuitively, without formal instruction or explanation of purpose or use (e.g., number lines, dots, tallies, base* 10 *blocks).*	b) Represent or describe rational numbers or numerical relationships using number lines and diagrams. *+For example, an item might require completion of a representation to show that a number and the opposite of the number are the same distance from* 0 *on a number line.*	

Exhibit 2.2. Number Properties and Operations (continued)

Num – 1. Number sense (continued)		
Grade 4	**Grade 8**	**Grade 12**
c) Compose or decompose whole quantities either by place value (e.g., write whole numbers in expanded notation using place value: $342 = 300 + 40 + 2$ or $3 \times 100 + 4 \times 10 + 2 \times 1$) or convenience (e.g., to compute 4×27 decompose 27 into $25 + 2$ because 4×25 is 100, and 4×2 is 8 so 4×27 is 108). *^Items should limit numbers to whole numbers through* 999,999. *+Emphasis should be on numbers through* 999. *+Emphasis should be on application of place value concepts as a way to express quantities.*		
d) Write or rename whole numbers (e.g., 10: $5 + 5$, $12 – 2$, 2×5). *+Items should limit numbers to whole numbers through* 999,999. *+Emphasis should be on numbers through* 999. *+Emphasis should be on multiple representations of a number using different operations.*	d) Write or rename rational numbers. *+For example, an item might involve writing a fraction as a decimal or a decimal as a fraction.* *+Decimals can be terminating or repeating.*	# d) Represent, interpret, or compare expressions for real numbers, including expressions using exponents and *logarithms. *^For example, an item might include expressions containing* π *or the square root of* 2*, or numerical relationships represented on a number line or with a diagram.* *^Exponents can be negative or fractional.*

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra with statistics).
# Grade 12 objectives that provide opportunities for questions in mathematical literacy.

Exhibit 2.2. Number Properties and Operations (continued)

Num – 1. Number sense (continued)		
Grade 4	**Grade 8**	**Grade 12**
e) Connect across various representations for whole numbers, fractions, and decimals (e.g., number word, number symbol, visual representations). *+Items should involve representations that students can use intuitively, without formal instruction or explanation of purpose or use (e.g., number lines, dots, tallies, base* 10 *blocks).* *+For example, an item might include representation of a number on a number line or with an area diagram.*	e) Recognize, translate, or apply multiple representations of rational numbers (fractions, decimals, and percents) in meaningful contexts. *+Items should avoid renaming of rational numbers as described in Number Properties and Operations objective 1.d.* *+For example, an item might situate a representation or multiple representations in context, such as a thermometer in a temperature-related item or a fuel gauge in a gas-related item.*	
	f) Express or interpret large numbers using scientific notation from real-life contexts. *+Items should present a number as a quantity or measurement.*	# f) Represent or interpret expressions involving very large or very small numbers in scientific notation. *^Exponents can be negative.* *^Include items that require interpreting calculator or computer displays given in scientific notation.*
	g) Find absolute values or apply them to problem situations. *+For example, an item might ask for the locations of a number and the absolute value of the number on a number line.* *+Include items that use absolute value to represent distance.*	g) Represent, interpret, or compare expressions or problem situations involving absolute values.

Grade 12 objectives that provide opportunities for questions in mathematical literacy.

Exhibit 2.2. Number Properties and Operations (continued)

Num – 1. Number sense (continued)		
Grade 4	**Grade 8**	**Grade 12**
h) Recognize and generate simple equivalent (equal) fractions and explain why they are equivalent (e.g., by using drawings). *+Items should limit denominators of fractions to* 2, 3, 4, 5, 6, 8, 10, 12, *or* 100.	h) Order or compare rational numbers (fractions, decimals, percents, or integers) using various representations (e.g., number line). *+Include items that present values to be ordered or compared as quantities in familiar contexts.*	
i) Order or compare whole numbers, decimals, or fractions using common denominators or benchmarks. *+Items should involve ordering or comparing numbers of the same type (i.e., whole numbers, decimals, fractions), and limit numbers to:* • *whole numbers through* 999,999*;* • *fractions with denominators* 2, 3, 4, 5, 6, 8, 10, 12*, or* 100*; or* • *decimals to hundredths.*	i) Order or compare rational numbers including very large and small integers, and decimals and fractions close to zero. *+Include items that present one or more numbers in scientific notation.*	# i) Order or compare rational or irrational numbers, including very large and very small real numbers.

Grade 12 objectives that provide opportunities for questions in mathematical literacy.

Exhibit 2.2. Number Properties and Operations (continued)

Num – 2. Estimation		
Grade 4	**Grade 8**	**Grade 12**
a) Use benchmarks (well-known numbers used as meaningful points for comparison) for whole numbers, decimals, or fractions in contexts (e.g., ½ and 0.5 may be used as benchmarks for fractions and decimals between 0 and 1.00). *+Items should limit benchmarks to numbers of the same type, using fraction benchmarks for fractions and decimal benchmarks for decimals.*	a) Establish or apply benchmarks for rational numbers and common irrational numbers (e.g., π) in contexts. *+Items can involve minimal context provided for the purpose of determining an appropriate benchmark.*	
b) Make estimates appropriate to a given situation with whole numbers, fractions, or decimals. *+Items should limit numbers to:* • *whole numbers through* 999,999*;* • *fractions with denominators* 2, 3, 4, 5, 6, 8, 10, 12, *or* 100*; or* • *decimals to hundredths.*	b) Make estimates appropriate to a given situation by: • Identifying when estimation is appropriate, • Determining the level of accuracy needed, • Selecting the appropriate method of estimation. *+Items should avoid estimation of square and cube roots as described in Number Properties and Operations objective 2.d.*	# b) Identify situations where estimation is appropriate, determine the needed degree of accuracy, and *analyze the effect of the estimation method on the accuracy of results. *+Items should avoid estimation of square and cube roots as described in Number Properties and Operations objective 2.d.*

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra with statistics).
# Grade 12 objectives that provide opportunities for questions in mathematical literacy.

Exhibit 2.2. Number Properties and Operations (continued)

Num – 2. Estimation (continued)		
Grade 4	**Grade 8**	**Grade 12**
c) Verify and defend solutions or determine the reasonableness of results in meaningful contexts. *+Items should avoid estimation as described in Number Properties and Operations objective 2.b.* *+For example, an item might require justification for a whole number response based on the context used in division involving a remainder.*	c) Verify solutions or determine the reasonableness of results in a variety of situations, including calculator or computer results. *+Item should focus on solutions to and results from real-world and mathematical situations appropriate for grade 8 (e.g., determining the reasonableness of a calculation involving a whole number exponent).* *+Items should avoid estimation as described in Number Properties and Operations objectives 2.b and 2.d.*	# c) Verify solutions or determine the reasonableness of results in a variety of situations. *+Items should avoid estimation as described in Number Properties and Operations objectives 2.b and 2.d.* *^Include items that involve using estimation and order of magnitude to determine the reasonableness of technology-aided computations and interpreting results in terms of the context (e.g., verifying a computation involving numbers written in scientific notation).*
	d) Estimate square or cube roots of numbers less than 150 between two whole numbers. *^Items should limit numbers to whole numbers between perfect squares* 1 *through* 144 *or perfect cubes* 1 *through* 125. *+Items should not allow use of a calculator.*	d) Estimate square or cube roots of numbers less than 1,000 between two whole numbers. *+Items should limit numbers to whole numbers between perfect squares* 1 *through* 900 *or perfect cubes* 1 *through* 729. *+Items should not allow use of a calculator.*

Grade 12 objectives that provide opportunities for questions in mathematical literacy.

Exhibit 2.2. Number Properties and Operations (continued)

Num – 3. Number operations		
Grade 4	**Grade 8**	**Grade 12**
a) Add and subtract using conventional or unconventional procedures (e.g., strategic decomposing and composing): • Whole numbers, or • Fractions and mixed numbers with like denominators. *+Items should limit numbers to whole numbers through* 9,999 *or fractions with denominators* 2, 3, 4, 5, 6, 8, 10, 12, *or* 100. *+Items that use a mathematical context should not allow use of a calculator.* *^Include items using a mathematical context that require computation with common fractions.*	a) Perform computations with rational numbers. *+Items that use a mathematical context should not allow use of a calculator.* *Include items that:* *^use a mathematical context and require computation with common and decimal fractions.* *^use a real-world context.* *+require recognition of a numerical expression equivalent to a given numerical expression that allows for a friendlier computation (e.g., adding up to solve fraction subtraction problems, doubling and halving to solve fraction multiplication problems).* *+require selection or creation of representations of a rational number computation (e.g., representing rational number division when the quotient is not a whole number).*	a) Find integer or simple rational powers of real numbers. *+Items that use a mathematical context should not allow use of a calculator.* *^For example, an item might require the evaluation of* $27^{1/3}$. *^Include items that involve numbers expressed with negative exponents.*

Exhibit 2.2. Number Properties and Operations (continued)

Num – 3. Number operations (continued)		
Grade 4	**Grade 8**	**Grade 12**
b) Multiply numbers using conventional or unconventional procedures (e.g., strategic decomposing and composing): • Whole numbers no larger than two digits by two digits with paper and pencil computation, or • Larger whole numbers using a calculator, or • Multiplying a fraction by a whole number. *+Items presenting unconventional procedures should focus on an efficient procedure for multiplying based on the given factors.* *+Items should limit denominators of fractions to* 1, 2, 3, 4, 5, 6, 8, 10, 12, *or* 100. *^Multiplication problems involving decimal fractions (e.g., money) can be included on calculator blocks.*		b) Perform arithmetic operations with real numbers, including common irrational numbers. *^Items should not include absolute value, which is addressed in Number Properties and Operations objective 3.c.* *+Items that use a mathematical context should not allow use of a calculator.* *^Include items that:* • *use a mathematical context and require computation with common and decimal fractions.* • *use a real-world context.* • *require application of order of operations.*
c) Divide whole numbers: • Up to three digits by one digit with paper and pencil computation, or • Up to five digits by two digits with use of calculator. *Items written for calculator blocks should not have remainders.*		c) Perform arithmetic operations with expressions involving absolute value.

Exhibit 2.2. Number Properties and Operations (continued)

<table>
<tr><th colspan="3">Num – 3. Number operations (continued)</th></tr>
<tr><th>Grade 4</th><th>Grade 8</th><th>Grade 12</th></tr>
<tr><td></td><td>d) Describe the effect of operations on size, including the effect of attempts to multiply or divide a rational number by:
• Zero, or
• A number less than zero, or
• A number between zero and one, or
• One, or
• A number greater than one.

^For example, an item might ask about the effect of multiplying a fraction by a fraction less than one, or a fraction by a fraction greater than one.</td><td>d) Describe the effect of multiplying and dividing by numbers including the effect of attempts to multiply or divide a real number by:
• Zero, or
• A number less than zero, or
• A number between zero and one, or
• One, or
• A number greater than one.

^For example, an item might ask about the effect of multiplying $2\sqrt{3}$ *by* 1/2.</td></tr>
<tr><td>e) Interpret, explain, or justify whole number operations and explain the relationships between them.

^Emphasis should be on interpreting, explaining, or justifying:
• *subtracting a number as the inverse operation to adding a number, or*
• *dividing by a number as the inverse operation to multiplying a number.*</td><td>e) Interpret, explain, or justify rational number operations and explain the relationships between them.

^Emphasis should be on interpreting, explaining, or justifying:
• *the four operations (including additive and multiplicative inverses),*
• *whole number square roots,*
• *whole number cube roots, or*
• *integer exponents.*</td><td>e) *Analyze or interpret a proof by mathematical induction of a simple numerical relationship.

+For example, an item might require proving that the sum of consecutive whole numbers from 0 *to n can be determined using the expression* $n(n+1)/2$.</td></tr>
</table>

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra with statistics).

Exhibit 2.2. Number Properties and Operations (continued)

Num – 3. Number operations (continued)		
Grade 4	**Grade 8**	**Grade 12**
f) Solve problems involving whole numbers and fractions with like denominators. *+Items should avoid concepts assessed by Measurement objectives, such as determining the perimeter of a rectangle.* *+Include items that present contexts using a variety of addition/ subtraction problem structures (e.g., add to, take from, put together/take apart, compare) and multiplication/ division problem structures (e.g., equal groups, arrays, area, compare).* *+Include items that require no more than three unique mathematical operations (addition, subtraction, multiplication, or division).* *^See Number Properties and Operations objectives 3.a, 3.b, and 3.c for number limitations and computation specifications.*	f) Solve problems involving rational numbers and operations using exact answers or estimates as appropriate. *+Items should avoid concepts assessed by Measurement or Geometry objectives, such as determining the volume of a cube.*	# f) Solve problems involving numbers, including rationals and common irrationals. *+Items should avoid concepts assessed by Measurement or Geometry objectives, such as application of the Pythagorean Theorem or determining the volume of a cylinder.*

Grade 12 objectives that provide opportunities for questions in mathematical literacy.

Exhibit 2.2. Number Properties and Operations (continued)

Num – 4. Ratios and proportional reasoning		
Grade 4	**Grade 8**	**Grade 12**
	a) Use ratios to describe problem situations. *+A ratio can be written a/b, a : b, or a to b.*	
	b) Use fractions to represent and express ratios and proportions. *+Include items that involve:* • *ratios of whole numbers* • *ratios of fractions*	
	c) Use proportional reasoning to model and solve problems (including rates and scaling). *+Items should avoid scale drawings, which are addressed in Measurement objective 2.f.*	# c) Use proportions to solve problems (including rates of change and per capita problems). *^Items should avoid scale drawings, which are addressed in Measurement objective 2.f.*
	d) Solve problems involving percentages (including percent increase and decrease, interest rates, tax, discount, tips, or part/whole relationships).	# d) Solve multistep problems involving percentages, including compound percentages.

Grade 12 objectives that provide opportunities for questions in mathematical literacy.

Exhibit 2.2. Number Properties and Operations (continued)

Num – 5. Properties of number and operations		
Grade 4	**Grade 8**	**Grade 12**
a) Identify odd and even numbers. *+Include items that involve determining whether the number of objects in a given set is even or odd.* *+Include items that involve writing an even number as the sum of two equal addends or as a sum of twos.*		
b) Identify factors of whole numbers. *^Items should involve identification of single-digit factors of whole numbers through* 100.	b) Recognize, find, or use factors, multiples, or prime factorization. *^Items should involve lowest common multiple, greatest common factor, or common multiples.* *^Items written for noncalculator blocks should use numbers less than* 400. *^Items written for calculator blocks should use numbers less than* 1,000.	
	c) Recognize or use prime and composite numbers to solve problems. *+Items can use a mathematical context or a real-world context.*	c) Solve problems using factors, multiples, or prime factorization. *+Items can use a mathematical context or a real-world context.* *^Include items that involve prime numbers.*
	d) Use divisibility or remainders in problem settings. *+Items should use a real-world context.* *+Items at grade 8 should be less complex than those developed at grade 12 (e.g., involve rational numbers).*	# d) Use divisibility or remainders in problem settings. *+Items should use a real-world context.* *+Items at grade 12 should be relevant to older students and may be more complex than those at grade 8 (e.g., involve irrational numbers).*

Grade 12 objectives that provide opportunities for questions in mathematical literacy.

Exhibit 2.2. Number Properties and Operations (continued)

Num – 5. Properties of number and operations (continued)		
Grade 4	**Grade 8**	**Grade 12**
e) Apply basic properties of operations. *^Items should involve the commutative and associative properties of addition and multiplication, the distributive property of multiplication across addition, the identity property of addition, and multiplication by zero.* *+Items should not assess naming of properties.* *+Emphasis should be on properties rather than computation.* *+See Number Properties and Operations objectives 3.a and 3.b for number limitations and computation specifications.*	e) Apply basic properties of operations, including conventions about the order of operations as applied to integers and rational numbers. *^Items should involve the commutative and associative properties of addition and multiplication, the distributive property of multiplication across addition, the identity and inverse properties of addition and multiplication, and multiplication by zero.* *+Items should not assess naming of properties.* *+Emphasis should be on properties rather than computation with rational numbers.*	e) Apply basic properties of operations, including conventions about the order of operations as applied to real numbers. *^Items should involve the commutative and associative properties of addition and multiplication, the distributive property of multiplication across addition, the identity and inverse properties of addition and multiplication, and multiplication by zero.* *+Items should not assess naming of properties.* *^Emphasis should be on properties rather than computation with real numbers, including irrational numbers.*
		f) Recognize properties of the number system (whole numbers, integers, rational numbers, real numbers, and *complex numbers) and how they are related to each other and identify examples of each type of number. *^Items can include questions about identifying irrational numbers (e.g., Which number is irrational:* 0.333, 0.333…, 3.14, √3*?).*

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra with statistics).

Measurement

Measuring is the process by which numbers are assigned to describe the world quantitatively. This process involves selecting the attribute of the object or event to be measured, comparing this attribute to a unit, and reporting the number of units. For example, in measuring a banner, one may select the attribute of length and the inch as a unit for the comparison. In comparing lengths to the nearest inch, it may be that a length is *about* 42 inches. If considering only the domain of whole numbers, one would report that the banner is 42 inches long. However, because length is a continuous attribute, in the domain of rational numbers the length of the banner might be reported as $41^{13}/_{16}$ inches (to the nearest 16^{th} of an inch) or 41.8 inches (to the nearest 0.1 inch).

The connection between measuring and number makes measurement a vital part of school mathematics. Measurement is an important setting for negative and irrational numbers as well as positive numbers, since negative numbers arise naturally from situations with two directions and irrational numbers are commonplace in geometry. Measurement representations and tools are often used when students are learning about number properties and operations. For example, area grids and representations of volume using unit cubes can help students understand multiplication and its properties. The number line can help students understand ordering and rounding numbers. Measurement also has a strong connection to other areas of school mathematics and other subjects. Problems in algebra are often drawn from measurement situations and functions are used to relate measures to each other. Geometry regularly focuses on measurement aspects of geometric figures. Probability and statistics provide ways to measure chance and to compare sets of data. The measurement of time, values of goods and services, physical properties of objects, distances, and various kinds of rates exemplify the importance of measurement in everyday activities.

In the Framework, attributes such as capacity, weight, mass, time, and temperature are included, as are the geometric attributes of length, area, and volume. Many of these attributes appear in grade 4, where the emphasis is on length, including perimeter, distance, and height. At grade 4, students do not use formulas to determine area. Instead, they use informal or physical understandings (e.g., grids or blocks). More emphasis is placed on area and angle measure in grade 8. By grade 12, measurement in everyday life, as well as in the study of volumes and rates constructed from other attributes, such as speed, is emphasized.

The 2026 NAEP Mathematics Assessment includes nonstandard, customary, and metric units. At grade 4, common customary units such as inch, quart, pound, hour, and degree (for measuring angles) are included, and common metric units such as centimeter, liter, and gram are emphasized. Grades 8 and 12 include the use of both square and cubic units for measuring area, surface area, and volume; continued use of degrees for measuring angles; and constructed units such as miles per hour. Converting from one unit in a system to another, such as from minutes to hours, is an important aspect of measurement included in problem situations. Understanding and using the many conversions available is an important skill. There are a limited number of common, everyday equivalencies that students are expected to know. These are described in the General Guidelines for Measurement subsection of this chapter.

Items classified in this content area depend on some knowledge of measurement. For example, an item comparing a 2-foot segment with an 8-inch line segment is classified as a measurement item, whereas an item that asks for the difference between a 3-inch and a 1¾-inch line segment would be classified as a number item. In many secondary schools, measurement becomes an integral part of geometry, and this is reflected in the proportion of items recommended for these two areas (see Exhibit 2.1).

The items in Illustrations 2.5 and 2.6 demonstrate the difference between a number item that involves units of measure and a measurement item. In the grade 4 item in Illustration 2.5, the context of weight is not necessary to determine the two consecutive whole numbers between which 12.4 lies. Since the focus of the item is comparing values, the item assesses a Number Properties and Operations objective.

Illustration 2.5. Example: A Number Properties and Operations Item Involving Units of Measure

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
4	Number Properties and Operations	Other	Num – 1.i	SR – MC

A bag of potatoes weighs 12.4 pounds. Which of the following statements is true?

A. There are between 1 and 2 pounds of potatoes in the bag.

B. There are between 12 and 13 pounds of potatoes in the bag.

C. There are between 124 and 125 pounds of potatoes in the bag.

D. There are between 1,246 and 1,247 pounds of potatoes in the bag.

Scoring Information	
Key	B. There are between 12 and 13 pounds of potatoes in the bag.

The item in this illustration is based on a NAEP item. The original version of this item appeared in the 2013 grade 4 NAEP Mathematics Assessment with NAEP Item ID 2013-4M6 #3 M135801.

In the grade 12 NAEP released item in Illustration 2.6, a measurement context is the focus of the item. That is, the accuracy of the measurements used forms the foundation of the item and must be considered when determining the range of measurements for the area of the room. Therefore, this item assesses a Measurement objective.

Illustration 2.6. Example: An Item with a Measurement Focus

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
12	Measurement	Other	Meas – 2.e	SR – MC

Carlene told Kyle that a rectangular room measured 16 feet by 12 feet, to the nearest foot. This means that the length could measure between 15.5 feet and 16.5 feet and the width could measure between 11.5 feet and 12.5 feet.

Kyle performed the following calculations.

Dimensions (feet)	Area (square feet)
15 by 11	165
15.5 by 11.5	178.25
16 by 12	192
16.5 by 12.5	206.25
17 by 13	221

Of the following intervals, which is the smallest interval that contains all possible values of the area of the room?

A. Between 191.5 and 192.5 square feet

B. Between 191 and 193 square feet

C. Between 179 and 206 square feet

D. Between 178 and 207 square feet

E. Between 165 and 221 square feet

Scoring Information	
Key	D. Between 178 and 207 square feet

The item in this illustration is based on a NAEP item. The original version of this item appeared in the 2009 grade 12 NAEP Mathematics Assessment with NAEP Item ID 2009-12M2 #10 M176801.

General Guidelines for Measurement. This section describes specifications common to many of the measurement objectives. *Any attribute, unit, instrument, conversion factor, or formula included in a list at a lower grade level is also appropriate for a higher grade level.*

Attributes. Attributes used in items are cumulative and listed below.

- Grade 4: perimeter, height, distance, time, temperature, capacity, weight or mass, area, and angle measure. Item content should emphasize length (measures of length include measures of perimeter, height, and distance).
- Grade 8: all attributes listed for grade 4, surface area, and volume. Item content should emphasize area. Attributes such as speed, measured in terms of the attributes of time and distance, are also appropriate.
- Grade 12: all attributes listed for grades 4 and 8. Item content should emphasize area, surface area, and volume. Rates constructed from other attributes, such as speed or flow rate, are appropriate.

Units. Units used in items are cumulative and listed below.

- Grade 4: nonstandard units, common customary units (inch, foot, mile, cup, quart, gallon, pound, hour, minute, day, year, degrees of measured angles, degrees Fahrenheit), and common metric units (centimeter, millimeter, meter, liter, gram, degrees Celsius) for the allowed attributes at this grade level.
- Grade 8: all units listed for grade 4 and square units, cubic units, and constructed units such as miles per hour; additional customary units (yard, fluid ounce, pint, ounce, ton) and additional metric units (kilometer, kilogram) for the attributes at this grade level.
- Grade 12: all units listed for grades 4 and 8 for the attributes at this grade level.

Instruments. The following measurement instruments are commonly found in curricula. Variations based on the same principles could be used during item development (e.g., graduated cup measures).

- All grades: ruler, clock, thermometer, graduated cylinder, balance scales, scales, protractor.

Conversions. Equivalencies that should be known by students and not provided in items are cumulative and listed below. All other conversions should be provided.

- Grade 4: feet/inches, hours/minutes, and meters/centimeters.
- Grade 8: square and cubic unit conversions, common time equivalences, and all common metric equivalences.
- Grade 12: conversions involving constructed units such as miles per hour to feet per minute.

Formulas. Grade 4 students are not expected to know any measurement formulas; however, they are expected to know at least one method for determining the perimeter, and at least one method for determining the area, of a rectangle. That is, students are expected to know that the perimeter of a rectangle can be determined by adding the lengths of all of its sides, but they do not need to know the formula $P = 2l + 2w$. Additionally, students can determine the area of a rectangle by tiling it with unit squares, without gaps or overlap, then counting the number of unit squares, or by multiplying the length and the width, but they do not need to know the formula $A = l \bullet w$.

Both grade 8 and grade 12 students should know formulas for the areas of a rectangle, a triangle, and a circle; the circumference of a circle; and the volumes of a cylinder and a rectangular solid. When other formulas are needed to complete an item, they should be given. See the General Guidelines for Geometry subsection of this chapter for more information about formulas for area, circumference, and volume.

The 2026 Measurement objectives are shown in Exhibit 2.3. Included with many of the objectives is italicized text providing clarifications or limitations for use during item development.

Exhibit 2.3. Measurement (Meas)

Meas – 1. Measuring physical attributes		
Grade 4	**Grade 8**	**Grade 12**
a) Identify the attribute that is appropriate to measure in a given situation. *+See the General Guidelines for Measurement for clarifications and limitations on attributes used in items.*		
b) Compare objects with respect to a given attribute, such as length, area, capacity, time, or temperature. *+Items involving area should avoid computing areas as described by Measurement objective 1.g.* *+See the General Guidelines for Measurement for clarifications and limitations on attributes used in items.*	b) Compare objects with respect to length, area, volume, angle measurement, weight, or mass. *+See the General Guidelines for Measurement for clarifications and limitations on attributes used in items.*	# b) Determine the effect of proportions and scaling on length, area, and volume. *+See the General Guidelines for Measurement for clarifications and limitations on attributes and units used in items.*
c) Estimate the size of an object with respect to a given measurement attribute (e.g., length, perimeter, or area using a grid). *+For example, an item might require estimating the area of an irregular shape presented on a grid.* *+See the General Guidelines for Measurement for clarifications and limitations on attributes and units used in items.*	c) Estimate the size of an object with respect to a given measurement attribute (e.g., area). *+See the General Guidelines for Measurement for clarifications and limitations on attributes and units used in items.*	# c) Estimate or compare perimeters or areas of two-dimensional geometric figures. *+See the General Guidelines for Measurement for clarifications and limitations on units used in items.*

Grade 12 objectives that provide opportunities for questions in mathematical literacy.

Exhibit 2.3. Measurement (continued)

Meas – 1. Measuring physical attributes (continued)		
Grade 4	**Grade 8**	**Grade 12**
		d) Solve problems of angle measure, including those involving triangles or other polygons or parallel lines cut by a transversal. *^Items should assume that students know* • *that the sum of the measures of the interior angles of a triangle is* 180º, *and* • *the relationships among the measures of angles formed by parallel lines cut by a transversal.*
e) Select or use appropriate measurement instruments such as ruler, meter stick, clock, thermometer, or other scaled instruments. *+"Other scaled instruments" may include a protractor.* *+See the General Guidelines for Measurement for clarifications on measurement instruments used in items.*	e) Select or use appropriate measurement instruments to determine or create a given length, area, volume, angle, weight, or mass. *+See the General Guidelines for Measurement for clarifications on measurement instruments used in items.*	
f) Solve problems involving perimeter of plane figures. *+Plane figures can be polygons but cannot be circles.* *+See the General Guidelines for Measurement for clarifications and limitations on units used in items.*	f) Solve mathematical or real-world problems involving perimeter or area of plane figures such as triangles, rectangles, circles, or composite figures. *+See the General Guidelines for Measurement for clarifications and limitations on units used in items.*	f) Solve problems involving perimeter or area of plane figures such as polygons, circles, or composite figures. *+See the General Guidelines for Measurement for clarifications and limitations on units used in items.*

Exhibit 2.3. Measurement (continued)

Meas – 1. Measuring physical attributes (continued)		
Grade 4	**Grade 8**	**Grade 12**
g) Solve problems involving area of squares and rectangles. *+Items should use measurements and right-angle markings, as appropriate, when art includes squares or rectangles.* *+Items should not require a formula but should assume that students know at least one method for determining the area of a square or rectangle.* *+Include items that relate area to the operations of multiplication and addition, such as tiling a rectangle with whole number side lengths and showing that the area is the same as would be found by multiplying the side lengths.* *+See the General Guidelines for Measurement for clarifications and limitations on units used in items.*		
	h) Solve problems involving volume or surface area of rectangular solids, and volume of right cylinders and prisms, or composite shapes. *+See the General Guidelines for Measurement for clarifications and limitations on units used in items.* *+See the General Guidelines for Geometry for clarifications and limitations on the formulas that items should assume students know or can use.*	# h) Solve problems by determining, estimating, or comparing volumes or surface areas of three-dimensional figures. *+See the General Guidelines for Measurement for clarifications and limitations on units used in items.* *+See the General Guidelines for Geometry for clarifications and limitations on the formulas that items should assume students know or can use.*
	i) Solve problems involving rates and ratios such as speed or population density. *+See the General Guidelines for Measurement for clarifications and limitations on units used in items.*	# i) Solve problems involving rates and ratios such as speed, density, population density, or flow rates. *+See the General Guidelines for Measurement for clarifications and limitations on units used in items.*

Grade 12 objectives that provide opportunities for questions in mathematical literacy.

Exhibit 2.3. Measurement (continued)

Meas – 2. Systems of measurement		
Grade 4	**Grade 8**	**Grade 12**
a) Select or use an appropriate type of unit for the attribute being measured such as length, angle size, time, or temperature. *+See the General Guidelines for Measurement for clarifications and limitations on attributes and units used in items.*	a) Select or use an appropriate type of unit for the attribute being measured such as length, area, angle, time, or volume. *+See the General Guidelines for Measurement for clarifications and limitations on attributes and units used in items.*	# a) Choose appropriate units for geometric measurements (length, area, perimeter, volume) and apply units in expressions, equations, and problem solutions. *+See the General Guidelines for Measurement for clarifications and limitations on attributes and units used in items.*
b) Solve problems involving conversions within the same measurement system such as conversions involving inches and feet or hours and minutes. *+Emphasis should be on conversions of measurements from a larger unit to a smaller unit.* *^Items can include additional conversions given the conversion information (e.g.,* 1 *quart =* 2 *pints).* *+See the General Guidelines for Measurement for conversions that should be known and not provided.*	b) Solve problems involving conversions within the same measurement system such as conversions involving square inches and square feet. *+See the General Guidelines for Measurement for conversions that should be known and not provided.*	# b) Solve problems involving conversions within or between measurement systems, given a relationship between the units. *^Conversions can include cubic units and compound rates such as miles per hour to feet per second.* *^See the General Guidelines for Measurement for conversions that should be known and not provided.*
	c) Estimate the measure of an object in one system given the measure of that object in another system and the approximate conversion factor. For example: • Distance: 1 kilometer is approximately 0.6 mile. • Money: U.S. dollars to Canadian dollars. • Temperature: Fahrenheit to Celsius.	

Grade 12 objectives that provide opportunities for questions in mathematical literacy.

Exhibit 2.3. Measurement (continued)

Meas – 2. Systems of measurement (continued)		
Grade 4	**Grade 8**	**Grade 12**
d) Determine appropriate unit of measurement in problem situations involving such attributes as length, time, capacity, or weight. *+See the General Guidelines for Measurement for clarifications and limitations on attributes and units used in items.*	d) Determine appropriate unit of measurement in problem situations involving such attributes as length, area, or volume. *+See the General Guidelines for Measurement for clarifications and limitations on attributes and units used in items.*	# d) Understand that numerical values associated with measurements of physical quantities are approximate, subject to variation, and must be assigned units of measurement. *+See the General Guidelines for Measurement for limitations on units used in items.*
		# e) Determine appropriate accuracy of measurement in problem situations (e.g., the accuracy of measurement of the dimensions to obtain a specified accuracy of area) and find the measure to that degree of accuracy. *^For example, an item might ask for the range within which the actual area of a rectangle could be if the side lengths of the rectangle measured to the nearest inch are 3 inches and 5 inches.*
	f) Construct or solve problems (e.g., floor area of a room) involving scale drawings. *+Include items that involve:* • *computing actual lengths and areas from a scale drawing* • *reproducing a scale drawing at a different scale*	# f) Construct or solve problems involving scale drawings. *^For example, an item might require determination of the number of rolls of insulation needed for insulating a house.* *^A scale drawing can be excluded from the item stem.*

Grade 12 objectives that provide opportunities for questions in mathematical literacy.

Exhibit 2.3. Measurement (continued)

Meas – 3. Measurement in triangles		
Grade 4	**Grade 8**	**Grade 12**
		# a) Solve problems involving indirect measurement. *^For example, an item might require determining the height of a building using the distance to the base of the building and the angle of elevation to the top of the building.*
		b) Solve problems using the fact that trigonometric ratios (sine, cosine, and tangent) stay constant in similar triangles. *^For example, an item might ask why the tangents of corresponding angles of two similar triangles are equal.*
		c) Use the definitions of sine, cosine, and tangent as ratios of sides in a right triangle to solve problems about length of sides and measure of angles. *^Items should assume that students know* • *the definitions of sine, cosine, and tangent, and* • *the side relationships for triangles with angle measurements of* 45-45-90 *and* 30-60-90.
		d) * Interpret and use the identity $\sin^2\theta + \cos^2\theta = 1$ for angles θ between 0° and 90°; recognize this identity as a special representation of the Pythagorean theorem. *^Items should assume that students know that* $\sin^2\theta + \cos^2\theta = 1$.

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra with statistics).
# Grade 12 objectives that provide opportunities for questions in mathematical literacy.

Exhibit 2.3. Measurement (continued)

Meas – 3. Measurement in triangles (continued)		
Grade 4	**Grade 8**	**Grade 12**
		e) * Determine the radian measure of an angle and explain how radian measurement is related to a circle of radius 1. *^Items should limit angle measures to* π/6, π/4, π/3, π/2, *and angles in other quadrants with these same referent angles.*
		f) * Use trigonometric formulas such as addition and double angle formulas. *^Items should provide relevant trigonometric formulas (e.g., law of cosines, double-angle formula).* *^For example, an item might require an explanation for whether or not* $\sin 20^\circ$ *and* $2 \sin 10^\circ$ *are equivalent.*
		g) * Use the law of cosines and the law of sines to find unknown sides and angles of a triangle. *^Items should provide relevant trigonometric formulas (e.g., law of cosines, double-angle formula).*
		h) * Interpret the graphs of the sine, cosine, and tangent functions with respect to periodicity and values of these functions for multiples of π/6 and π/4.

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra with statistics).

Geometry

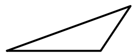
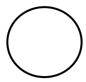
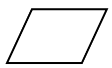
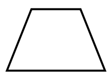
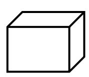
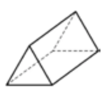
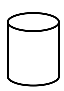
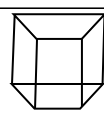
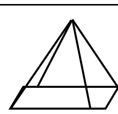
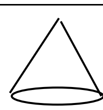
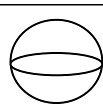
Geometry began thousands of years ago in many lands as sets of practical rules related to describing and predicting locations of astronomical objects, calculating land areas, and building structures. More than 2,200 years ago, the Greek mathematician Euclid organized the geometry known at that time into a coherent collection of results, all deduced using logic from a small number of postulates assumed to be true. Euclid's work was fundamental in establishing mathematical truth as dependent on valid deductive reasoning rather than reliant on educated guesses from several specific examples. The theorems obtained via deduction by Euclid remain fundamental to the study of geometry, and for this reason the geometry studied in school is called Euclidean geometry.

The fundamental concepts of Euclidean geometry are congruence, similarity, and symmetry. By grade 4, students are expected to be familiar with a library of simple figures and their attributes, both in the plane (lines, circles, triangles, squares, and rectangles) and in space (cubes, spheres, and cylinders).

By grade 8, understanding of these shapes deepens, with study of cross sections of solids and the beginnings of an analytical understanding of properties of plane figures, especially parallelism, perpendicularity, and angle relations in polygons. Reflections, translations, and rotations (mathematical models of the physical phenomena of reflecting, sliding, and turning) are introduced as distance-preserving transformations that map a figure onto a congruent image. Dilatations (expansions and contractions) map figures onto similar images. Properties of congruent and similar figures involve angle measures and lengths, so geometry becomes more and more mixed with measurement in later grades. Placing figures on a coordinate plane provides the beginnings of the connections among algebra, geometry, and analytic geometry.

In secondary school, the content of plane geometry is logically ordered, and students are expected to make, test, and validate conjectures. Students see that most of the commonly studied plane figures—triangles (scalene, isosceles, equilateral) and quadrilaterals (parallelogram, rectangle, rhombus, square, trapezoid)—may possess reflection or rotation symmetry, or both, and can use triangle congruence and similarity theorems as well as symmetry to establish properties of figures. By grade 12, students may also gain insight into systematic structure, such as the classification of distance-preserving transformations of the plane (that is, reflections, rotations, translations, or glide reflections), and what happens when two or more isometries are performed in succession (composition). In analytic geometry, the key areas of geometry and algebra merge into a powerful tool that provides a basis for calculus and much of applied mathematics.

General Guidelines for Geometry. This table provides expectations for knowledge of geometric formulas at each grade level.

Shape	Formulas for Area and Circumference		
	Grade 4	Grade 8	Grade 12
Rectangle 	find area without a formula	expected to know the formula	expected to know the formula
Triangle 	not tested	expected to know the formula	expected to know the formula
Circle 	not tested	expected to know the formula	expected to know the formula
Parallelogram 	not tested	formula should be provided	formula should be provided
Trapezoid 	not tested	formula should be provided	formula should be provided
Figure	**Formulas for Volume and Surface Area**		
	Grade 4	**Grade 8**	**Grade 12**
Rectangular prism 	not tested	expected to know the formula	expected to know the formula
Triangular prism 	not tested	expected to know the formula	expected to know the formula
Right circular cylinder 	not tested	expected to know the formula for volume only	expected to know the formula
General prisms 	not tested	formula should be provided	formula should be provided
Square pyramid 	not tested	formula should be provided	formula should be provided
Right circular cone 	not tested	formula should be provided	formula should be provided
Sphere 	not tested	formula should be provided	formula should be provided

The 2026 Geometry objectives are shown in Exhibit 2.4. Included with many of the objectives is italicized text providing clarifications or limitations for use during item development.

Exhibit 2.4. Geometry (Geom)

Geom – 1. Dimension and shape		
Grade 4	**Grade 8**	**Grade 12**
a) Identify or describe (informally) real-world objects using simple plane figures (e.g., triangles, rectangles, squares, and circles) and simple solid figures (e.g., cubes, spheres, and cylinders). *^For example, an item might require identification of rectangles in a picture of a room.*	a) Identify a geometric object given a written description of its properties. *+Items should include geometric objects appropriate to grade 8, such as polygons, composite shapes, and right pyramids, prisms, and cones.*	
b) Identify or draw angles and other geometric figures in the plane. *+Geometric figures can include points, lines, line segments, rays, polygons, and circles.*	b) Identify, define, or describe geometric shapes in the plane and in three-dimensional space given a visual representation. *^Items should be more complex than those presented at grade 4, such as those involving geometric shapes and figures composed of triangles, quadrilaterals, polygons, cubes, or right prisms.*	b) Give precise mathematical descriptions or definitions of geometric shapes in the plane and in three-dimensional space. *^Three-dimensional shapes can include the full set of Platonic solids (e.g., cube, regular tetrahedron).*
	c) Draw or sketch from a written description polygons, circles, or semicircles.	c) Draw or sketch from a written description plane figures and planar images of three-dimensional figures. *Figures can include isosceles triangles, regular polygons, polyhedra, spheres, and hemispheres.*

Exhibit 2.4. Geometry (continued)

Geom – 1. Dimension and shape (continued)		
Grade 4	**Grade 8**	**Grade 12**
		# d) Use two-dimensional representations of three-dimensional objects to visualize and solve problems. *+Items should involve three-dimensional objects composed of triangles, rectangles, and/or circles (e.g., net of a cylinder in a context about packages of oatmeal).*
e) Describe or distinguish among attributes of two- and three-dimensional shapes. *+Items should focus on countable or defining attributes, such as number of sides or number of right angles, and should avoid concepts assessed by Measurement objectives, such as determining perimeter or area.* *+For example, an item might require identification of characteristics that all rectangles have in common.*	e) Demonstrate an understanding of two- and three-dimensional shapes in the world through identifying, drawing, reasoning from visual representations, composing, or decomposing. *+For example, an item might involve use of a cylinder to represent a construction barrel, or recognition that a cube can be decomposed into four same-sized pyramids or three noncongruent pyramids having equal volumes.*	# e) Analyze properties of three-dimensional figures including prisms, pyramids, cylinders, cones, spheres, and hemispheres. *+Items should avoid explicitly requiring the volume or surface area of a prism, pyramid, cylinder, cone, sphere, or hemisphere, but may require analysis of a familiar object to determine if it has properties similar to one of the named figures.* *+For example, an item might require an informal argument for the formula for the volume of a cylinder, the volume of a pyramid, or the volume of a cone.*

Grade 12 objectives that provide opportunities for questions in mathematical literacy.

Exhibit 2.4. Geometry (continued)

Geom – 2. Transformation of figures and preservation of properties		
Grade 4	**Grade 8**	**Grade 12**
	a) Identify lines of symmetry in plane figures or recognize and classify types of symmetries of plane figures. *^Items should involve point, line, and rotational symmetry.*	a) Recognize or identify types of symmetries (e.g., translation, reflection, rotation) of two- and three-dimensional figures.
		b) Give or recognize the precise mathematical relationship (e.g., congruence, similarity, orientation) between a figure and its image under a transformation. *Transformations can include reflections, rotations, translations, and dilations.*
	c) Recognize or informally describe the effect of a transformation (reflection, rotation, translation, or dilation) on two-dimensional figures. *+For example, an item might require recognition that any transformation takes a line segment to a line segment, but that the type of transformation determines whether the line segments have the same length.*	c) Perform or describe the effect of a single transformation (reflection, rotation, translation, or dilation) on two- or three-dimensional geometric figures. *+Items can involve more than one application of a single type of transformation (e.g., viewing of the image of a reflection of an image in a mirror).*
d) Recognize attributes (such as shape and area) that do not change when plane figures are subdivided and rearranged. *+Items should limit plane figures to those composed of triangles and rectangles.* *+Items can involve subdividing while maintaining the original shape.*	d) Predict results of combining, subdividing, and recombining shapes of plane figures and solids (e.g., paper folding, tiling, subdividing and rearranging the pieces).	d) Identify transformations of shapes that preserve the area of two-dimensional figures or the volume of three-dimensional figures. *Items can include the comparison of the areas of two different shapes.*

Exhibit 2.4. Geometry (continued)

Geom – 2. Transformation of figures and preservation of properties (continued)		
Grade 4	**Grade 8**	**Grade 12**
	e) Justify relationships of congruence and similarity and apply these relationships using scaling and proportional reasoning. *^Items should limit figures to those in two dimensions.*	e) Justify relationships of congruence and similarity and apply these relationships using scaling, proportional reasoning, and established theorems. *^Items should allow for a variety of forms of proof (e.g., flow diagram, paragraph, two-column).* *^Proofs can include standard SAS, SSS, or ASA congruence proofs with corresponding parts.* *^Include items that* • *apply scaling and proportional reasoning to two-dimensional figures.* • *apply scaling and proportional reasoning to three-dimensional figures.* • *ask for justifications less formal than proofs of established theorems (e.g., giving reasons why figures are congruent or similar).*
	f) Apply the relationships among angle measures, lengths, and perimeters among similar figures. *^Emphasis should be on right triangles and quadrilaterals.*	f) Apply the relationships among angle measures, lengths, perimeters, and volumes among similar figures. *+For example, an item might present two similar triangles with the necessary measures and require determining a missing angle measure or side length in one of the triangles.*

Exhibit 2.4. Geometry (continued)

Geom – 2. Transformation of figures and preservation of properties (continued)		
Grade 4	**Grade 8**	**Grade 12**
		g) Perform or describe the effects of successive (composites of) isometries and/or similarity transformations. *+Items should be limited to transformations on one-dimensional geometric objects, two-dimensional geometric shapes, or three-dimensional geometric figures.* *+Items should avoid transformations on algebraic representations as described in Algebra objective 2.d.* *+For example, an item might require the selection of a different set of transformations that have the same result as a series of three reflections over three parallel lines.*

Exhibit 2.4. Geometry (continued)

Geom – 3. Relationships between geometric figures		
Grade 4	**Grade 8**	**Grade 12**
a) Analyze or describe patterns in polygons when the number of sides increases, or the size or orientation changes.		
b) Combine simple plane shapes to construct a given shape. *+Include items that involve combining two-dimensional shapes to construct a three-dimensional figure.*	b) Apply geometric properties and relationships in solving problems in two and three dimensions. *^Items should limit figures to parallel and perpendicular lines, triangles, quadrilaterals, circles, cylinders, and cones.* *^Include items that involve properties of geometric similarity, congruence, and angle sum.* *^Include items that involve angle relationships and transversal properties of quadrilateral angles.*	b) Apply geometric properties and relationships to solve problems in two and three dimensions. *+Items should avoid concepts assessed by Measurement objectives, such as determining the volume of a composite figure.* *^Emphasis should be on solving problems.* *Problems can involve multiple steps.* *^Figures can include parallel and perpendicular lines, triangles (including triangles with angle measures of* 45-45-90 *and* 30-60-90*), cylinders, cones, prisms, and pyramids.*
c) Recognize two-dimensional faces of three-dimensional shapes.	c) Represent problem situations with geometric figures to solve problems. *+Emphasis should be on grade-level appropriate representations or figures.* *+Include items that involve solving real-world problems.*	# c) Represent problem situations with geometric figures to solve problems. *+Items should be more complex than grade 8 items. For example, grade 12 items might involve more figures, or more properties, than grade 8 items.* *^Emphasis should be on representations or figures.*

Grade 12 objectives that provide opportunities for questions in mathematical literacy.

Exhibit 2.4. Geometry (continued)

Geom – 3. Relationships between geometric figures (continued)		
Grade 4	**Grade 8**	**Grade 12**
	d) Use the Pythagorean theorem to solve problems in two-dimensional situations. *^Items should assume that students know the Pythagorean theorem.* *+Items can use a real-world context.* *+Include items that involve application of the Pythagorean Theorem to determine the distance between two points.*	# d) Use the Pythagorean theorem to solve problems in two- or three-dimensional situations. *^Items should assume that students know the Pythagorean theorem.*
		e) Recall and interpret or use definitions and basic properties of congruent and similar triangles, quadrilaterals, and other polygons; circles; parallel, perpendicular, and intersecting lines; and associated angle relationships (e.g., in solving problems or creating proofs). *^Emphasis should be on direct application of definitions or defining properties of lines, angles, and shapes.*
f) Describe and compare attributes of simple and compound figures composed of triangles, squares, and rectangles. *+For example, an item might provide a rectangular prism and require identification of the faces that have the same area.*	f) Describe, compare, or analyze attributes of, or relationships between, triangles, quadrilaterals, and other polygonal plane figures. *+Items should avoid situations in which the definition of a trapezoid must be assumed.*	f) Analyze attributes or relationships of triangles, quadrilaterals, and other polygonal plane figures. *^Items should avoid situations in which the definition of a trapezoid must be assumed.* *^Emphasis should be on examining figures, identifying their properties, and applying identified properties.* *^Figures can include rhombi, parallelograms, and trapezoids.*

Grade 12 objectives that provide opportunities for questions in mathematical literacy.

Exhibit 2.4. Geometry (continued)

Geom – 3. Relationships between geometric figures (continued)		
Grade 4	**Grade 8**	**Grade 12**
	g) Describe or analyze properties and relationships of parallel or intersecting lines. *+For example, an item might present a pair of parallel lines cut by a transversal and require identification of the angles that have the same measure.*	g) Analyze properties and relationships of parallel, perpendicular, or intersecting lines, including the angle relationships that arise in these cases. *^Emphasis should be on examining lines and angles, identifying their properties, and applying identified properties.*
		h) Make, test, and validate geometric conjectures using a variety of methods, including deductive reasoning and counterexamples.
		i) * Analyze properties of circles and the intersections of lines and circles (inscribed angles, central angles, tangents, secants, and chords). *^For example, an item might ask about measures of angles inscribed in a semicircle, or the relationships among tangents, secants, chords, and radii.*

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra with statistics).

Exhibit 2.4. Geometry (continued)

Geom – 4. Position, direction, and coordinate geometry		
Grade 4	**Grade 8**	**Grade 12**
a) Describe relative positions of points and lines using the geometric ideas of parallelism or perpendicularity.	a) Describe relative positions of points and lines using the geometric ideas of midpoint, points on a common line through a common point, parallelism, or perpendicularity.	a) Solve problems involving the coordinate plane using distance between two points, the midpoint of a segment, or slopes of perpendicular or parallel lines. *+Items should avoid application of the Pythagorean theorem as described in Geometry objective 3.d.* *+Items should avoid concepts assessed by Algebra objectives, such as determining the equation of a line through two points.* *+For example, an item might involve determining the slope of a line given two points or given the slope of a line to which it is perpendicular.*
	b) Describe the intersection of two or more geometric figures in the plane (e.g., intersection of a circle and a line).	b) Describe the intersections of lines in the plane and in space, of a line and a plane, or of two planes in space.
	c) Visualize or describe the cross section of a solid. *^Items should involve cross sections of standard, familiar solids such as a sphere, cylinder, or rectangular solid.*	c) Describe or identify conic sections and other cross sections of solids. *^Items should involve cross sections of standard, familiar solids such as a cone, sphere, or cylinder, and of Platonic solids such as a cube or regular tetrahedron.*
	d) Represent geometric figures using rectangular coordinates on a plane.	d) Represent two-dimensional figures algebraically using coordinates and/or equations.
		e) * Use vectors to represent velocity and direction; multiply a vector by a scalar and add vectors both algebraically and graphically.

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra with statistics).

Exhibit 2.4. Geometry (continued)

Geom – 4. Position, direction, and coordinate geometry (continued)		
Grade 4	**Grade 8**	**Grade 12**
		f) Find an equation of a circle given its center and radius and, given an equation of a circle, find its center and radius. *^Items should assume that students know the equation of a circle.* *^Include items that require the derivation of the center or radius of a circle.*
		g) * Graph or determine equations for images of lines, circles, parabolas, and other curves under translations and reflections in the coordinate plane. *^Items should provide the formulas for ellipses and hyperbolas in standard form.* *^Items should not require knowledge of technical characteristics of these functions (e.g., equations of asymptotes or foci).* *^Items can require knowledge of general characteristics of these functions (e.g., drawing a graph).*
		h) * Represent situations and solve problems involving polar coordinates.

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra with statistics).

Data Analysis, Statistics, and Probability

Data analysis and statistics refers to the entire process of collecting, organizing, summarizing, and interpreting data. This is the heart of statistics and is in evidence whenever quantitative information is used to determine a course of action. Data analysis normally begins with a question to be answered. Statistical questions can arise prior to data collection, or from existing data sets. Beginning at an early age, students should grasp the fundamental principle that exploratory data analysis of an existing data set is far different from the scientific method of collecting data to verify or refute a well-posed question. Data can be useful when collected with a specific question in mind and when there is a plan (usually called a *design*) for using the data to answer the question. However, contemporary uses of data-mining techniques associated with "big data" suggest that data sets may subsequently be useful in answering questions that were not envisioned when the data collection was initiated.

A probability is a measure of uncertainty. This measure may be determined from a theoretical model that makes assumptions about equally likely or weighted outcomes for an event (as when one says that the probability of a coin landing head-side up is one-half) or it may be determined in some way from past experience, as when forecasters say the probability of rain tomorrow is 40 percent. Statistical analysis often involves studying whether assumptions about theoretical probability match observed relative frequencies. For instance, if a coin tossed 100 times turned up heads 80 times, one might suspect that the probability of heads for that coin is not ½ (the theoretical probability of heads for a fair coin). Under random sampling, patterns for outcomes of designed studies can be anticipated and used as a basis for making decisions. The probability distribution of all possible outcomes is important in most statistical decision-making because the key is to decide whether or not a particular observed outcome is typical or unusual (located in a tail of a probability distribution). For example, 4.0 as a grade-point average is unusually high among most student groups, 4 as the weight in pounds of a human baby is unusually low, and 4 as the number of floors in a building is not unusual in either direction.

By grade 4, students are expected to apply their understanding of number and quantity to consider questions that can be answered by examining appropriate data. Building on the principles of describing data distributions through minimum, maximum, and clusters of values, grade 8 students are expected to use a wider variety of organizing and summarizing techniques for center, spread, and shape. They can identify and construct a statistical question, one that needs data in order to be addressed. They can also begin to analyze statistical claims through designed surveys and experiments that involve randomization. Also by grade 8, students are expected to begin to use more formal terminology related to probability and data analysis. They can identify associations between two numerical variables in scatterplots, as well as the relative strength of those associations.

Grade 12 students are expected to use a wide variety of statistical techniques for all phases of data analysis, including a more formal understanding of statistical inference, and simulation as an inferential analysis tool. In addition to comparing univariate data sets, students at this level can recognize and describe possible associations between two variables by looking at two-way tables for categorical variables or scatterplots for measurement variables. By grade 12, students should be able to use linear equations to describe possible associations between measurement variables and should be familiar with techniques for fitting functions to data.

Implications of Updates to Data Analysis, Statistics, and Probability Objectives. As mentioned in Chapter 1, a re-examination of statistics, data analysis, and probability concepts and skills in light of current scholarship and content of standards documents led to significant changes in the objectives for this content area at grade 4. Along with the decrease in the number of Data Analysis, Statistics, and Probability objectives, the phrasing of objectives has changed. Illustration 2.7 compares wording for an objective in grade 4 that was revised.

Illustration 2.7. Grade 4 Data Analysis, Statistics, and Probability Objective 2.b

Objective	2017 Wording	2026 Wording
2.b	Given a set of data or a graph, describe the distribution of data using median, range, or mode.	Given a distribution of whole number data in a context, identify and explain the meaning of the greatest value, of the least value, or of any clustering or grouping of data in the distribution.

The composite item in Illustration 2.8 shows two ways objective 2.b can be assessed in grade 4. The item is adapted from *Key Stage 2, Paper 3: Reasoning* (Standards and Testing Agency, 2019) and contains material developed by the Standards and Testing Agency for England's 2019 national curriculum assessment, licensed under Open Government Licence v3.0. Key Stage 2 students are 7 to 11 years old.

Illustration 2.8. Example: Item Aligning to Grade 4 Objective 2.b

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
4	Data Analysis, Statistics, and Probability	Other	Data – 2.b	SCR – composite

This chart shows the masses of eight kittens in grams (g).

305 g	375 g	310 g	255 g
275 g	410 g	360 g	345 g

A. Complete these sentences about the masses of the kittens.

The mass of the lightest kitten is [305 / 255 / 275 / 345] grams.

The mass of the heaviest kitten is [375 / 310 / 410 / 360] grams.

B. Complete this table to put the masses of the kittens into four groups.

Mass (grams)	Number of kittens
250 – 299	
300 – 349	
350 – 399	
400 – 449	

Scoring Information

Part A 255; 410

Part B

Mass (grams)	Number of kittens
250 – 299	**2**
300 – 349	**3**
350 – 399	**2**
400 – 449	**1**

The item in this illustration is adapted from an England *Key Stage 2* item. The original version of this item appeared as Item 7 in the 2019 administration of *Paper 3: Reasoning*.

General Guidelines for Data Analysis, Statistics, and Probability. This section describes additional specifications for data representations used in items at each grade level.

- Limitations on representations of data for each grade level are indicated in the fourth row of Exhibit 2.5. Within an objective, a parenthetical list of representations indicates which of the grade-level appropriate representations is applicable for that objective.
- Items should include interpretation of a variety of less common representations of data, such as those found in newspapers and magazines.
- Bar graphs and plots over time (line graphs) should increase in complexity (e.g., through using more complex scales and greater numbers of categories) from grade to grade.
- Descriptions of data sets at grade 4 may be informal.

The 2026 Data Analysis, Statistics, and Probability objectives are shown in Exhibit 2.5. Included with many of the objectives is italicized text providing clarifications or limitations for use during item development.

Exhibit 2.5. Data Analysis, Statistics, and Probability (Data)

<table>
<tr><th colspan="3">Data – 1. Data representation</th></tr>
<tr><th>Grade 4</th><th>Grade 8</th><th>Grade 12</th></tr>
<tr><td colspan="3">Representations of data are indicated for each grade level in the next row. For some objectives, only a subset of the representations is applicable, indicated by a parenthetical list at the end of the objective.</td></tr>
<tr><td>Pictographs, bar graphs, dot plots, tables, and tallies.</td><td>Histograms, plots over time, dot plots, scatterplots, box plots, bar graphs, circle graphs, stem and leaf plots, frequency distributions, and tables.</td><td>Histograms, plots over time, dot plots, scatterplots, box plots, bar graphs, circle graphs, stem and leaf plots, frequency distributions, and tables, including two-way tables.</td></tr>
<tr><td>a) Read or interpret a single distribution of data.

+Representations of data can be graphical or tabular.</td><td>a) Read or interpret data, including interpolating or extrapolating from data.

+Representations of data can be graphical or tabular.</td><td># a) Read or interpret graphical or tabular representations of data.</td></tr>
<tr><td>b) For a given distribution of data, complete a graph (limits of time make it difficult to construct graphs completely).</td><td>b) For a given distribution of data, complete a graph and solve a problem using the data in the graph (histograms, plots over time, dot plots, scatterplots, bar graphs, circle graphs).</td><td># b) For a given set of data, complete a graph and solve a problem using the data in the graph (histograms, plots over time, dot plots, scatterplots).

+Items should involve a single data set and a single data representation.</td></tr>
<tr><td>c) Answer statistical questions by estimating and computing within a single distribution of data.</td><td>c) Answer statistical questions by estimating and computing with data from a single distribution or across distributions of data.</td><td>c) Answer statistical questions involving univariate or bivariate distributions of data.

+Items can utilize any of the representations listed for grade 12.

^Include items that require using multiple sets of data. For example, an item might require construction and comparison of three box plots based on given data sets.</td></tr>
</table>

Grade 12 objectives that provide opportunities for questions in mathematical literacy.

Exhibit 2.5. Data Analysis, Statistics, and Probability (continued)

Data – 1. Data representation		
Grade 4	**Grade 8**	**Grade 12**
	d) Given a graphical or tabular representation of a distribution of data, determine whether the information is represented effectively and appropriately (histograms, plots over time, dot plots, scatterplots, box plots, bar graphs, circle graphs).	# d) Analyze, compare, and contrast different graphical representations of univariate and bivariate data (e.g., identify misleading uses of data in real-world settings and critique different ways of presenting and using information). *^For example, an item might ask for a comparison of the effects of scale changes on the representation of data in a graph.*
		# e) * Organize and display data in a spreadsheet in order to recognize patterns and solve problems. *^Items can ask for the manipulation of spreadsheets, the recognition of patterns displayed in a spreadsheet, or the use of data to solve problems.*

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra with statistics).

\# Grade 12 objectives that provide opportunities for questions in mathematical literacy.

Exhibit 2.5. Data Analysis, Statistics, and Probability (continued)

Data – 2. Characteristics of data sets		
Grade 4	**Grade 8**	**Grade 12**
	a) Calculate, use, or interpret mean, median, mode, range, or shape of a distribution of data.	# a) Calculate, interpret, or use summary statistics for distributions of data including measures of center (mean, median), position (quartiles, percentiles), spread (range, interquartile range, variance, and standard deviation) or shape (skew, uniform, uni-/bimodal). *+Items involving shape should focus on interpreting and using.*
b) Given a distribution of whole number data in a context, identify and explain the meaning of the greatest value, of the least value, or of any clustering or grouping of data in the distribution. *+The terms "clustering" and "grouping" can be used interchangeably but should not both be used in the same item.* *+Include items that allow students to describe clustering/grouping of data within a distribution.*	b) Describe a distribution of data using its mean, median, mode, range, interquartile range, and shape.	b) Recognize how linear transformations of one-variable data affect mean, median, mode, range, interquartile range, and standard deviation. *^For example, an item might ask about the effect on the mean when a constant is added to each data point in a set.*
	c) Identify outliers and determine their effect on the mean, median, mode, or range.	# c) Determine the effect of outliers on the mean, median, mode, range, interquartile range, or standard deviation.
	d) Using appropriate statistical measures, compare two or more data sets describing the same characteristic for two different populations or subsets of the same population. *^Items should limit statistical measures to mean, median, mode, range, and interquartile range.*	# d) Compare data sets using summary statistics (mean, median, mode, range, interquartile range, shape, or standard deviation) describing the same characteristic for two different populations or subsets of the same population.

Grade 12 objectives that provide opportunities for questions in mathematical literacy.

Exhibit 2.5. Data Analysis, Statistics, and Probability (continued)

Data – 2. Characteristics of data sets (continued)		
Grade 4	**Grade 8**	**Grade 12**
	e) Visually choose the line that best fits given a scatterplot and informally explain the meaning of the line. Use the line to make predictions. *+Items should present a scatterplot but not require drawing a line of best fit on the scatterplot.*	e) Approximate a trend line if a linear pattern is apparent in a scatterplot or use a graphing calculator to determine a least-squares regression line and use the line or equation to make predictions. *^Items can require the use of technology to construct a least-squares regression line from a small data set.*
		# f) Recognize or explain how an argument based on data might confuse correlation with causation. *+For example, an item might require the critique of an argument about one of two strongly correlated variables causing change in the other.*
		g) * Identify and interpret the key characteristics of a normal distribution such as shape, center (mean), and spread (standard deviation).
		# h) * Recognize and explain the potential errors that can arise when extrapolating from data. *^For example, an item might require an explanation of the danger of using a line of best fit to make predictions for values well beyond the range of the given data.*

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra with statistics).
# Grade 12 objectives that provide opportunities for questions in mathematical literacy.

Exhibit 2.5. Data Analysis, Statistics, and Probability (continued)

Data – 3. Experiments and samples		
Grade 4	**Grade 8**	**Grade 12**
	a) Given a sample, identify possible sources of bias in sampling. *+For example, an item might require identification of whether the members of a sample are representative of the population of interest.*	# a) Identify possible sources of bias in sample survey populations or questions and describe how such bias can be controlled and reduced.
	b) Distinguish between a random sample and a nonrandom sample.	b) Recognize and describe a method to select a simple random sample. *+Items should focus on ways to select a random sample where every element of the population has the same likelihood of being selected.* *+Items should not assess the impact of random sampling on bias as described in Data Analysis, Statistics, and Probability objective 3.a.* *+For example, an item might involve using a random number generator to model a population.*
		# c) Draw inferences from samples, such as estimates of proportions in a population, estimates of population means, or decisions about differences in means for two “treatments.”
		d) Identify or evaluate the characteristics of a good survey or of a well-designed experiment. *+For example, an item might require reasoning about whether a sample is of sufficient size to draw conclusions about the population of interest.*

Grade 12 objectives that provide opportunities for questions in mathematical literacy.

Exhibit 2.5. Data Analysis, Statistics, and Probability (continued)

Data – 3. Experiments and samples (continued)		
Grade 4	**Grade 8**	**Grade 12**
		e) * Recognize the differences in design and in conclusions between randomized experiments and observational studies. *^For example, an item might ask about different sources of bias between the two types of studies, how randomness is considered in each type, or how changes in variables are treated.*

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra with statistics).

Exhibit 2.5. Data Analysis, Statistics, and Probability (continued)

Data – 4. Probability		
Grade 4	**Grade 8**	**Grade 12**
		# a) Determine whether two events are independent or dependent.
	b) Using assumption of randomness, determine the theoretical probability of simple or compound events in familiar contexts. *^Items should use familiar contexts such as rolling a number cube, flipping a coin, or spinning the arrow of a spinner.*	# b) Using assumptions such as randomness, determine the theoretical probability of simple or compound events in familiar or unfamiliar contexts. *^Items should use* • *simple events that are independent or dependent, or* • *compound events that are independent.*
	c) Given the results of an experiment or simulation, estimate the probability of simple and compound events in familiar contexts. *^Items should use familiar contexts such as rolling a number cube, flipping a coin, or spinning the arrow of a spinner.*	# c) Given the results of an experiment or simulation, estimate the probability of simple or compound events in familiar or unfamiliar contexts. *^For example, an item might require an explanation involving how the relative frequency of occurrences of a specified outcome is not the same as its probability but can be used to estimate the probability of the outcome (e.g., Anita flipped a coin* 10 *times and got 7 heads, but the theoretical probability of a head is not* 0.7*).*

Grade 12 objectives that provide opportunities for questions in mathematical literacy.

Exhibit 2.5. Data Analysis, Statistics, and Probability (continued)

Data – 4. Probability (continued)		
Grade 4	**Grade 8**	**Grade 12**
	d) Use theoretical probability to evaluate or predict experimental outcomes in familiar contexts. *+Items should use familiar contexts such as rolling a number cube, flipping a coin, or spinning the arrow of a spinner.*	# d) Use theoretical probability to evaluate or predict experimental outcomes in familiar or unfamiliar contexts. *^Items should be more complex than those at grade 8 (e.g., involve more events).* *+Item should present contexts of interest to a large cross section of students. To increase the likelihood of capturing interests of the assessed students, the item pool should include a variety of student-relevant contexts.*
	e) Determine the sample space for a given situation. *+Include items that allow students to determine the number of different ways in which objects can be grouped (e.g., given three shirts and two pairs of pants, show how to determine the number of ways the shirts and pants can be paired).*	e) Determine the number of ways an event can occur using tree diagrams, formulas for combinations and permutations, or other counting techniques. *^Items should assess understanding of how to generate sample spaces.*
	f) Use a sample space to determine the probability of possible outcomes for an event.	
	g) Represent the probability of a given outcome using fractions, decimals, and percents. *+Items should involve writing a description of an outcome as a probability and should not involve calculating probabilities.*	
	h) Determine the probability of independent and dependent events. (Dependent events should be limited to a small sample size.)	h) Determine the probability of independent and dependent events. *^Items should use simple events that are independent or dependent, or compound events that are dependent.*

Grade 12 objectives that provide opportunities for questions in mathematical literacy.

Exhibit 2.5. Data Analysis, Statistics, and Probability (continued)

Data – 4. Probability (continued)		
Grade 4	**Grade 8**	**Grade 12**
		i) Determine conditional probability using two-way tables.
	j) Interpret and apply probability concepts to practical situations, and simple games of chance. *+Items should* • *assume that students are not familiar with specifics regarding playing cards, such as the number of cards in a deck, the suits represented in a deck of cards, or the number of cards of each suit;* • *use "number cube" instead of "dice" and assume that students are not familiar with the specifics of a number cube, such as the numbers presented on each face; and* • *avoid references to gambling.* *+For example, an item might state that* 10% *of the population is left-handed and require an estimate of the number of students that are left-handed in a school with* 825 *students.*	# j) Interpret and apply probability concepts to practical situations, including odds of success or failure in simple lotteries or games of chance. *+Items should* • *assume that students are not familiar with specifics regarding playing cards, such as the number of cards in a deck, the suits represented in a deck of cards, or the number of cards of each suit;* • *use "number cube" instead of "dice" and assume that students are not familiar with the specifics of a number cube, such as the numbers presented on each face; and* • *avoid references to gambling.*
		k) * Use the binomial theorem to solve problems. *^Items should provide the binomial theorem.* *^For example, an item might present a binomial problem situation with the probability of an event being* 0.1 *and require determination of the probability of that event occurring* 3 *out of* 11 *times.*

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra with statistics).

Grade 12 objectives that provide opportunities for questions in mathematical literacy.

Algebra

Algebra began in the use of systematic methods for solving problems and numerical puzzles by mathematicians in the Middle East, South Asia, and China, and made its way to Europe in the late Middle Ages. The modern symbolic notation, with letters to stand for unknowns and constants, was developed in the 16^{th} century. The notation so greatly enhanced the power of the algebraic method that the basic ideas of both analytic geometry and calculus were developed within a century.

The increased use of algebra led to study of its formal structure. Gradually, the "rules of algebra" were distilled into a compact summary of the principles behind algebraic manipulation. In the 19^{th} century, these principles (e.g., commutativity, distributivity) were codified into a deductive system parallel to that of Euclidean geometry. A corresponding line of thought produced a simple but flexible concept of function and also led to the development of set theory as a comprehensive background for mathematics. When taken broadly as including these ideas, the study and uses of algebra reach from the foundations of mathematics to the frontiers of current research.

The notion of variable—a symbol that can stand for any member of an identified set—has multiple facets (e.g., as an unknown, parameter, or varying quantity); variables are used in many ways in school mathematics. Variables are used to express structural generalizations such as the commutativity of addition. In formulas such as $d = rt$ or $c = \sqrt{a^2+b^2}$, variables stand for quantities that may take on a variety of values. In problem solving, a variable may represent an unknown quantity. The study of functions includes attention to independent variables, dependent variables and parameters.

When students make abstractions and generalizations about numbers and operations in early arithmetic by attending to underlying structure, they are engaging in algebraic thinking even though the formalism of algebraic notation may not be evident. As students progress through the grades, they continue to engage in algebraic thinking and they add more algebraic formalism to their repertoire.

By grade 4, students are expected to recognize and extend simple numeric patterns as a foundation for a later understanding of function. They begin to understand the meaning of equality and some of its properties, as well as the idea of an as-yet-unknown quantity as a precursor to the concept of variable. They also begin to informally explore properties of operations, including how inverse operations can be used to simplify a computation or how numbers can be decomposed and recomposed for more efficient computational strategies.

As students move into grade 8, the ideas of variable, covariation (two or more quantities varying simultaneously), and function become more important. By using variables to describe patterns and solve simple equations, students become familiar with manipulating them. Representations of covariation in tables, verbal descriptions, symbolic descriptions, and graphs can combine to promote a flexible grasp of the idea of function. Linear functions receive special attention: they connect to the ideas of proportionality, ratio, and rate, forming a bridge that will eventually link arithmetic to calculus. Symbolic manipulation in the relatively simple context of linear equations is reinforced by other ways of finding solutions, including graphing by hand or with technology.

By grade 12, students are expected to be skillful at manipulating and interpreting more complex expressions. Nonlinear functions, especially quadratic, power, and exponential functions whose graphs are accessible using graphing technology, are used by students to solve real-world problems. Grade 12 students are also expected to be accomplished at translating verbal descriptions of problem situations into symbolic form. Also, by grade 12, students should understand expressions involving several variables, systems of linear equations, and solutions to inequalities.

General Guidelines for Algebra. Overall, items at grade 4 highlight informal algebra. For example, there is an emphasis on "completing number sentences" instead of "solving equations." At grade 8, items cover some formal algebra, but the expectation is that less formal algebra content will be included. For example, determining solutions of higher-degree polynomial equations or systems of linear or nonlinear equations is not expected at grade 8, but is expected at grade 12.

At grade 12, the types of functions eligible for use in all items are linear, quadratic, rational, exponential, and trigonometric. Rational functions are limited to those with a constant or linear numerator and a linear or quadratic denominator. Rational expressions are limited in the same way. Trigonometric functions are limited to sine, cosine, and tangent. Logarithmic functions can be used only in items written for objectives identified with an asterisk (*).

The 2026 Algebra objectives are shown in Exhibit 2.6. Included with many of the objectives is italicized text providing clarifications or limitations for use during item development.

Exhibit 2.6. Algebra (Alg)

Alg – 1. Patterns, relations, and functions		
Grade 4	**Grade 8**	**Grade 12**
a) Recognize, describe (in words or symbols), or extend simple numerical and visual patterns. *+Items should assess extensions of patterns in mathematically appropriate ways. For example, patterns should either be presented in ways that are transferable to a larger set or allow for multiple correct responses when not transferable to a larger set (e.g., when the first six elements of a pattern do not necessarily indicate the next six elements).* *+Pattern types can include whole numbers or shapes.*	a) Recognize, describe, or extend numerical and visual patterns using tables, graphs, words, or symbols. *^Items should involve more complex patterns than those presented at grade 4.* *+Items should assess extensions of patterns in mathematically appropriate ways. For example, patterns should either be presented in ways that are transferable to a larger set or allow for multiple correct responses when not transferable to a larger set (e.g., when the first six elements of a pattern do not necessarily indicate the next six elements).* *+Items should avoid linear patterns addressed by other Algebra objectives.* *^Pattern types can include rational numbers, powers, simple recursive patterns, regular polygons, and three-dimensional shapes.*	a) Recognize, describe, or extend numerical patterns, including arithmetic and geometric sequences (progressions). *^Items should clearly define the nature of the pattern in the problem.* *+Items should assess extensions of patterns in mathematically appropriate ways. For example, patterns should either be presented in ways that are transferable to a larger set or allow for multiple correct responses when not transferable to a larger set (e.g., when the first six elements of a pattern do not necessarily indicate the next six elements).* *+Items should avoid linear patterns addressed by other Algebra objectives.* *^Items can use patterns with multiple solutions when students are asked to explain their answers.* *^Pattern types can include those from grade 8, along with quadratic patterns and exponential patterns.* *Responses can include verbal descriptions or equations.*
		b) Express linear and exponential functions in recursive and explicit form given a verbal description, table, or some terms of a sequence. *^Include items that require* • *the explicit form of a function, given a recursive form.* • *the equation of a line, given a table of points.*

Exhibit 2.6. Algebra (continued)

<table>
<tr><th colspan="3">Alg – 1. Patterns, relations, and functions (continued)</th></tr>
<tr><th>Grade 4</th><th>Grade 8</th><th>Grade 12</th></tr>
<tr><td>c) Given a description, extend or find a missing term in a pattern or sequence.

+Items should involve rules that follow the clarifications and limitations of numbers and operations identified in the Number Properties and Operations objectives.</td><td>c) Examine or create patterns, sequences, or linear functions expressed as a rule numerically, verbally, or symbolically.</td><td></td></tr>
<tr><td>d) Create a different representation of a pattern or sequence given a verbal description.</td><td></td><td></td></tr>
<tr><td></td><td>e) Identify functions as linear or nonlinear or contrast distinguishing properties of functions from tables, graphs, or equations.

^Items can ask about properties of lines or curves, including slopes and intercepts, but determination of the value of the slope of a curve is not required.</td><td>e) Identify or analyze distinguishing properties of linear, quadratic, rational, exponential, or *trigonometric functions from tables, graphs, or equations.

+Items should avoid inverses as described in Algebra objective 2.j.

Items can include properties such as rate of change, intercepts, periodicity, or symmetry.</td></tr>
<tr><td></td><td>f) Interpret the meaning of slope or intercepts, or determine the rate of change between two points on a graph of a linear function.

+Items can use a real-world context.</td><td></td></tr>
<tr><td></td><td></td><td>g) Determine whether a relation, given in verbal, symbolic, tabular, or graphical form, is a function.</td></tr>
</table>

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra with statistics).

Exhibit 2.6. Algebra (continued)

Alg – 1. Patterns, relations, and functions (continued)		
Grade 4	**Grade 8**	**Grade 12**
		h) Recognize and analyze the general forms of linear, quadratic, rational, exponential, or *trigonometric functions. *+Items should avoid inverses as described in Algebra objective Alg – 2.j.* *^Items can include examining parameters and their effect on the graph of linear and quadratic functions (e.g., in $y = ax + b$, recognize the roles of a and b).*
		i) Determine the domain and range of functions given in various forms and contexts. *^Items should limit functions to linear, quadratic, inverse proportionality ($y = k/x$), absolute value, exponential, and trigonometric functions.* *^Items can include characteristics of domain and range in real-life contexts, or in functions such as $f(x) = \|x - 3\|$.*
		j) * Given a function, determine its inverse if it exists and explain the contextual meaning of the inverse for a given situation. *^For example, an item might ask: When $f(t)$ represents a population in year t, what is the meaning of $f^{-1}(3000) = 1965$?*

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra with statistics).

Exhibit 2.6. Algebra (continued)

Alg – 2. Algebraic representations		
Grade 4	**Grade 8**	**Grade 12**
a) Translate between different representational forms (symbolic, numerical, verbal, or pictorial) of whole number relationships (such as from a written description to an equation or from a function table to a written description). *+Items should involve whole number relationships that follow the clarifications and limitations of numbers and operations identified in the Number Properties and Operations objectives.*	a) Translate between different representations of linear expressions using symbols, graphs, tables, diagrams, or written descriptions.	a) Create and translate between different representations of algebraic expressions, equations, and inequalities (e.g., linear, quadratic, exponential, or *trigonometric) using symbols, graphs, tables, diagrams, or written descriptions. *^Items should require either* • *translating between two different forms of representation, or* • *given one form of representation, creating a different form of representation.* *Items can include those that require the construction of graphs.* *The stimulus can include symbols, graphs, tables, diagrams, or written descriptions.*
	b) Interpret and compare representations of linear relationships expressed in symbols, graphs, tables, diagrams, or written descriptions. *+Representations are limited to linear relationships.* *^Items can include identification of strengths and weaknesses of different representations for different purposes.*	# b) Interpret and compare representations of relationships expressed in symbols, graphs, tables, diagrams (including Venn diagrams), or written descriptions. *+Representations can include any linear or nonlinear relationship appropriate to grade 12.* *+Items can include identification of strengths and weaknesses of different representations for different purposes.*
	c) Graph or interpret points represented by ordered pairs of numbers on a rectangular coordinate system. *^Items should limit coordinates to rational numbers.*	

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra with statistics).
# Grade 12 objectives that provide opportunities for questions in mathematical literacy.

Exhibit 2.6. Algebra (continued)

Alg – 2. Algebraic representations (continued)		
Grade 4	**Grade 8**	**Grade 12**
	d) Solve problems involving coordinate pairs on the rectangular coordinate system. *^Items can include determining areas of simple geometric figures.*	d) Perform or interpret transformations on the graphs of linear, quadratic, exponential, and *trigonometric functions. *^Items should present the graph of the function in the stem.* *^For example, an item might ask for the vertex of the parabola resulting from $y = x^2$ being translated up 3 units and right 5 units, and then reflected over the line $y = x$.*
		e) Make inferences or predictions using an algebraic model of a situation.
	f) Identify or represent functional relationships in meaningful contexts including proportional, linear, and common nonlinear relationships (e.g., compound interest, bacterial growth) in tables, graphs, words, or symbols. *^Items involving nonlinear functions should have whole number powers.*	# f) Given a real-world situation, determine if a linear, quadratic, rational, exponential, *logarithmic, or *trigonometric function fits the situation. *^Examples of real-world situations can be projectile motion, half-life, bacterial growth, Richter scale for earthquakes, or logarithmic scales in graphs.*
		# g) Solve problems involving exponential growth and decay. *+Items can involve science or finance contexts that will be familiar to students. For example, an item might involve modeling the effect of remediation of exponential growth of the bacteria in spinach production.*

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra with statistics).

\# Grade 12 objectives that provide opportunities for questions in mathematical literacy.

Exhibit 2.6. Algebra (continued)

Alg – 2. Algebraic representations (continued)		
Grade 4	**Grade 8**	**Grade 12**
		h) *Identify distinguishing characteristics of exponential, logarithmic, and rational functions (e.g., discontinuity, asymptotes, concavity). *^Items should not require determining domains and ranges, which are addressed in Algebra objective 1.i.* *^Items can involve functions with points of discontinuity or asymptotes (vertical and horizontal).*

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra with statistics).

Exhibit 2.6. Algebra (continued)

Alg – 3. Variables, expressions, and operations		
Grade 4	**Grade 8**	**Grade 12**
a) Use letters and symbols to represent an unknown quantity in a simple mathematical expression. *+Items that involve numbers and operations should follow the clarifications and limitations identified in the Number Properties and Operations objectives.*		
b) Express simple mathematical relationships using expressions, equations, or inequalities.	b) Write algebraic expressions, equations, or inequalities to represent a situation. *^Items should limit expressions, equations, or inequalities to those with first degree terms.*	b) Write algebraic expressions, equations, or inequalities to represent a situation. *^Items can include determining the equation of a line given the slope and a point or given two points.* *^Expressions, equations, or inequalities can have terms of degree greater than one.*
	c) Perform basic operations, using appropriate tools, on linear algebraic expressions (including grouping and order of multiple operations involving basic operations, exponents, roots, simplifying, and expanding).	c) Perform basic operations, using appropriate tools, on algebraic expressions including polynomial and rational expressions.
		d) Write equivalent forms of algebraic expressions, equations, or inequalities to represent and explain mathematical relationships. *Items should address equivalent forms within one type of representation, not translating between different representations.*

Exhibit 2.6. Algebra (continued)

Alg – 3. Variables, expressions, and operations (continued)		
Grade 4	**Grade 8**	**Grade 12**
		# e) Evaluate algebraic expressions, including polynomials and rational expressions.
		f) Use function notation to evaluate a function at a specified point in its domain and combine functions by addition, subtraction, multiplication, division, and composition.
		g) * Determine the sum of finite and infinite arithmetic and geometric series. *^Items should provide formulas for the sum of a finite or infinite series.* *^For example, an item might ask for a range of possible total distances traveled by a ball when it is dropped from 20 feet above ground and makes three bounces, each up to 75% of its previous height.*
		h) Use basic properties of exponents and *logarithms to solve problems.

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra with statistics).

Grade 12 objectives that provide opportunities for questions in mathematical literacy.

Exhibit 2.6. Algebra (continued)

Alg – 4. Equations and inequalities		
Grade 4	**Grade 8**	**Grade 12**
a) Find the unknown(s) in a whole number sentence (e.g., in an equation or simple inequality like [_] + 3 > 7). *^Items should present equations and inequalities that involve no more than one operation in the process of determining an unknown or a set of unknowns.*	a) Solve linear equations or inequalities (e.g., Solve for x in $ax + b = c$ or $ax + b = cx + d$ or $ax + b > c$). *^Items in a noncalculator block should limit coefficients to rational numbers.*	a) Solve linear, rational, or quadratic equations or inequalities, including those involving absolute value. *^Items should assume that students know the quadratic formula.* *^Items should limit coefficients to real numbers.* *Items should not use complex roots.*
b) Interpret "=" as an equivalence between two values and use this interpretation to solve problems.		b) * Determine the role of hypotheses, logical implications, and conclusions in algebraic arguments about equality and inequality. *^For example, an item might require understanding that neither of the following statements can be reversed:* $y = x - 1$ *implies* $y^2 = (x - 1)^2$ *or* $f(x) = 0$ *implies* $g(x) \bullet f(x) = 0$.
c) Verify a conclusion using simple algebraic properties derived from work with numbers (e.g., commutativity, properties of 0 and 1). *^For example, an item might require understanding that if Sam is 3 years older than Ned, 20 years from now Sam will still be 3 years older than Ned.*	c) Make, validate, and justify conclusions and generalizations about linear relationships. *^Items should require inductive and deductive reasoning when recognizing, expressing, or using the connections among and between linear relationships.*	c) Use algebraic properties to develop a valid mathematical argument. *^Items should address properties of equality and properties of operations. For example, an item might require an explanation for why division by zero is undefined.*

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra with statistics).

Exhibit 2.6. Algebra (continued)

Alg – 4. Equations and inequalities (continued)		
Grade 4	**Grade 8**	**Grade 12**
	d) Analyze situations or solve problems using linear equations and inequalities with rational coefficients symbolically or graphically (e.g., $ax + b = c$ or $ax + b = cx + d$).	# d) Analyze situations, develop mathematical models, or solve problems using linear, quadratic, exponential, or *logarithmic equations or inequalities symbolically or graphically. *^Items should not involve complex roots.* *Items can include real number coefficients.*
	e) Interpret relationships between symbolic linear expressions and graphs of lines by identifying and computing slope and intercepts (e.g., in $y = ax + b$, know that a is the rate of change and b is the vertical intercept).	e) Solve (symbolically or graphically) a system of equations or inequalities and recognize the relationship between the analytical solution and graphical solution. *^Items should limit systems of equations to two linear equations or one linear equation and one quadratic equation.* *Items can assess compound inequalities.*
	f) Use and evaluate common formulas (e.g., relationship between a circle's circumference and diameter, $C = \pi d$, distance and time under constant speed). *^Items should utilize formulas that come from a familiar context or situation.*	# f) Solve problems involving special formulas such as: $A = P(I + r)^t$ or $A = Pe^{rt}$. *^Items should present special formulas and define all variables in presented special formulas.* *+For example, a mathematical literacy item might involve comparing amounts that would be paid back from loans of equal value but with different interest rates.*

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra with statistics).
# Grade 12 objectives that provide opportunities for questions in mathematical literacy.

Exhibit 2.6. Algebra (continued)

Alg – 4. Equations and inequalities (continued)		
Grade 4	**Grade 8**	**Grade 12**
		# g) Solve an equation or formula involving several variables for one variable in terms of the others. *+Items should assume that students know the quadratic formula.*
		h) * Solve quadratic equations with complex roots. *^Items should assume that students know the quadratic formula.*

* Objectives that describe mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra with statistics).
# Grade 12 objectives that provide opportunities for questions in mathematical literacy.

Revisions of the 2017 Content Objectives

Revisions to the 2017 NAEP mathematics content objectives resulted from consideration of a wide range of relevant sources. These included research on mathematical development and learning, each state's standards and frameworks for mathematics instruction and assessment in the United States, reviews of state standards in comparison to NAEP objectives (e.g., Johnston et al., 2018), research on the alignment between NAEP items and common standards (e.g., Daro, Hughes, & Stancavage, 2015), policy statements informing state standards (e.g., NCTM, 2000, 2014, 2018; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), *Guidelines for Assessment and Instruction in Statistics Education* (GAISE; Bargagliotti et al., 2020), *Guidelines for Assessment and Instruction in Mathematical Modeling Education* (GAIMME; Garfunkel & Montgomery, 2019), the content of leading international assessments (e.g., PISA [OECD, 2019] and TIMSS [NCES, 2019]), the professional judgment and experience of Panel members, and feedback obtained from readers of draft versions of the Framework.

Though overlapping, these sources were not in complete agreement regarding the mathematics students need to know and be able to do. Using this range of sources resulted in a set of objectives that cannot and will not be representative of what every child in the U.S. is taught by a given grade, nor will they conform precisely to the stated achievement objectives of any single state or professional organization. At the same time, the resulting objectives are tightly linked to acknowledged aspirations for the mathematics U.S. students should have an opportunity to learn. The content delineated here focuses on mathematical ideas that students are likely to have encountered in school.

Revisions attended to both current state standards—where the nation is now—and where the nation is likely headed. Updates to the content objectives were also motivated by several other considerations, including precision and accuracy of the language used to describe an objective; developmental appropriateness of objectives at a particular grade level, based on current research and state policies; and shifts in content emphases since the last framework update. In the case of a limited number of objectives that are not common in the majority of U.S. state standards, guidance came from the ways leading states and nations situate those topics in their respective content objectives.

Restructuring of "Mathematical Reasoning" as a Subtopic

Mathematical Reasoning subtopics appeared in the previous NAEP Mathematics Assessment Framework (Governing Board, 2017a) in Number Properties and Operations, Geometry, Data Analysis, Statistics, and Probability, and Algebra. With the introduction of the NAEP Mathematical Practices (see Chapter 3), most of the Mathematical Reasoning objectives will be measured by items aligned to a content objective and classified with one of the NAEP Mathematical Practices. To preserve attention to content that was uniquely present in some of the Mathematical Reasoning objectives, some content from those objectives was incorporated into other subtopics' objectives (e.g., Number and Operations subtopic 3.e in grades 4 and 8 was "Interpret…" and is now "Interpret, explain, or justify…").

Changes at Grade 4

In the early grades, up through grade 4, there is a distinction between NAEP content area arrangement and the arrangement common in many states' assessment standards. Most state assessments use three to five areas in the early grades, but these do not parallel the five areas used in NAEP. At the same time, it must be noted that analysis of state standards has indicated that some content in the previous objectives is now not regularly part of U.S. schooling until grade 5 or later (Daro et al., 2015; Hughes, Daro, Holtzman, & Middleton, 2013; Johnston et al., 2018). To address this, some objectives were removed at grade 4. In many cases, grade 8 objectives were similar and more appropriately timed to assess students on mathematics they would have had a chance to learn. Additionally, research comparing states' standards for curriculum and instruction with NAEP assessment objectives suggested that some content commonly taught by grade 4 was absent from NAEP (Johnston et al., 2018). Careful review of this analysis led to the modification or addition of objectives at grade 4. Research and development on the use of the equal sign as an equivalence between two values and its importance in the foundation for algebraic thinking (Carpenter, Franke, & Levi, 2003) has meant states include more attention to it. This greater attention led to the addition of one related objective in grade 4 Algebra. Increased work with certain concepts in early grades since the last NAEP Mathematics Framework update led to one addition and several modifications of grade 4 Number Properties and Operations objectives. Similarly, several grade 4 objectives in Data Analysis, Statistics, and Probability were modified to reflect current language use for noticing, using, and interpreting data.

Changes at Grade 8

Since the last NAEP framework update, there have been shifts in state standards in expectations about understanding and use of rates, recognition of pattern, and greater attention to data, statistics, and probability in grades 5, 6, 7, and 8 (i.e., after grade 4; Johnston et al., 2018). As a result, the grade 8 objectives in Data, Statistics, and Probability were revised to clarify expectations, and three grade 8 objectives were deleted because similar grade 4 objectives or grade 12 objectives were more appropriately timed to assess what students have an opportunity to learn.

Changes at Grade 12

At grade 12, as in the other grades, descriptions of objectives were edited to clarify measurement intent. Added in grade 12 were two objectives in Geometry and Measurement: one about periodicity of functions and one on applying geometric properties among similar figures in two and three dimensions. In some cases where an objective was identified as beyond what is commonly taught in grade 12, an asterisk (*) was added. Also, to support the possible reporting of Mathematical Literacy as a particular way in which students know and do mathematics at grade 12, a number sign (#) was added to indicate objectives relevant to the exploration of this reporting.

Changes in Item Distribution

As previously noted, the last decade has seen a shift of data and related topics to grades 5, 6, 7, and 8. Hence, the proportion of items for Data Analysis, Statistics, and Probability went up for grade 8 (from 15% to 20%) and down for grade 4 (from 10% to 5%). Concurrently, greater attention to fractions in grade 4 across states led to an increase in the proportion of Number Properties and Operations items (from 40% to 45%). Measurement in contexts that are not geometric play a smaller role in grade 8 than geometry topics, and the proportion of such items was reduced (from 15% to 10%). By grade 12, most new measurement ideas are in geometric contexts and, as in the previous NAEP Mathematics Framework, measurement and geometry continue to be treated together in the item distribution for grade 12. In fact, the distribution of items for each content area at grade 12 remains the same, reflecting the delineation of essential concepts in the literature on high school learning (NCTM, 2018).

NAEP MATHEMATICAL PRACTICES

Interest in students' mathematical practices has been growing for over 40 years. Seminal work by authors such as Collins and Stevens (1983), Lave (1988), Saxe (1988), and Schoenfeld (1985) focused on the cognitive skills and strategies used by mathematics experts and adults "in the wild" (i.e., outside of school). This line of research led to a distillation of the specific behaviors engaged during mathematical reasoning and problem solving, illuminating what are now called "practices" of mathematics.

Mathematics education research has also experienced a "social turn" (Lerman, 2000), marked by a shift toward investigating mathematics learning as it is situated in social activity, including discourse practices (Adler, 1999; Bell & Pape, 2012; Black, 2004; Civil & Planas, 2004; Enyedy, 2003; Ernest, 1998; Moschkovich, 2007, 2008; NCTM, 1991; van Oers, 2001). Students use their mathematical knowledge and skill in the social settings of school and home, on the basketball court, or in games they play with friends. The 2026 NAEP Mathematics Framework captures this broader and more complete picture of what it means to know and do mathematics. For the first time, NAEP Mathematics includes mathematical practices as a fundamental component of the assessment (see Exhibit 3.1). This chapter offers a brief overview of the research literature on mathematical practices as a whole and describes these five key NAEP Mathematical Practices in depth. As was the case with the content areas in Chapter 2, these five areas are not meant to be inclusive of all possible mathematical activity.

Exhibit 3.1. Summary of NAEP Mathematical Practices

NAEP Mathematical Practice 1: Representing

Recognizing, using, creating, interpreting, or translating among representations appropriate for the grade level and the mathematics being assessed.

NAEP Mathematical Practice 2: Abstracting and Generalizing

Decontextualizing, identifying commonality across cases, items, problems, or representations, and extending one's reasoning to a broader domain appropriate for the grade level and the mathematics being assessed.

NAEP Mathematical Practice 3: Justifying and Proving

Creating, evaluating, showing, or refuting mathematical claims in developmentally and mathematically appropriate ways.

NAEP Mathematical Practice 4: Mathematical Modeling

Making sense of a scenario, identifying a problem to be solved, mathematizing it, applying the mathematization to reach a solution, and checking the viability of the solution in developmentally and mathematically appropriate ways.

NAEP Mathematical Practice 5: Collaborative Mathematics

The social enterprise of doing mathematics with others through discussion and collaborative problem solving whereby ideas are offered, debated, connected, and built-upon toward solution and shared understanding. Collaborative mathematics involves joint thinking among individuals toward the construction of a problem solution in developmentally and mathematically appropriate ways.

Selecting Mathematical Practices for NAEP

The five NAEP Mathematical Practices are a particular distillation—for the purposes of assessment—of more than 40 years of research and development. They reflect a review of current scholarship, national and international assessment frameworks, national standards, and state standards more broadly.

To understand what mathematical practices are, it may be helpful to consider what they are not. Although practices underlie and contribute to mathematical reasoning, they are not completely synonymous with it, because many other skills contribute to mathematical reasoning, such as working memory (Geary, Hoard, Byrd-Craven, & DeSoto, 2004) and computational fluency (Geary, Liu, Chen, Saults, & Hoard, 1999). Similarly, although mathematical practices may contribute to conceptual understanding, the two are not interchangeable. On some accounts, conceptual understanding is knowledge of the underlying structure and relations represented in mathematics that transcends application of familiar algorithms (Eisenhart et al., 1993; Hiebert & Lefevre, 1986). In contrast, practices are fluid and responsive to both familiar and unfamiliar problems. Indeed, it is just as likely that conceptual understanding improves students' mathematical practices as it is that practices themselves improve conceptual understanding.

An increasing emphasis on mathematical practices is evident in state and national standards (NCTM, 1991, 2000, 2014). It is now generally agreed that knowing and doing mathematics entail engaging in practices such as generalizing, conjecturing, justifying, mathematizing, solving problems, communicating, and sense-making (Barbosa, 2006; Goos, 2004; Goos, Galbraith, & Renshaw, 2002; Hufferd-Ackles, Fuson, & Sherin, 2004; Hussain, Monaghan, & Threlfall, 2013; Lau, Singh, & Hwa, 2009; Truxaw & DeFranco, 2008). As students grapple with and discuss mathematical ideas and problems—individually and together—they engage in such mathematical practices, which serve to familiarize them with the norms of doing mathematics (Herbel-Eisenmann & Cirillo, 2009). The inclusion of NAEP Mathematical Practices is not separate from the mathematics content of Chapter 2. These practices are described separately to indicate the significant change to the NAEP Mathematics Framework in sufficient detail.

The term "mathematical practices" has been used by the field in a variety of ways, with state standards and NCTM standards offering two widely disseminated descriptions. Five specific practices have been selected for emphasis on the 2026 NAEP Mathematics Assessment; these are referred to throughout the Framework and the Assessment and Item Specifications as the NAEP Mathematical Practices. As further detailed in Chapter 4, the assessment is designed to measure content and practices together. However, not all items will include an assessed NAEP Mathematical Practice. In fact, not all NAEP content objectives need to be assessed alongside a NAEP Mathematical Practice. Some items will continue to assess content outside of the particular NAEP Mathematical Practices, such as items that focus on algorithms, procedural fluency, precision, tool use, or mathematical practices other than the five that are the focus for the NAEP Mathematics Assessment.

There are commonalities across the NAEP Mathematical Practices and the practices described in policy documents and common in state standards. For example, the NAEP Mathematical Practices and the NCTM Mathematical Process Standards include communication and collaboration, while communication is a subtext in several of the mathematical practices common in state standards (e.g., in critiquing the reasoning of others). Representing in the doing, teaching, and learning of mathematics is a process standard in NCTM's *Curriculum and Evaluation Standards for School Mathematics* (1989), *Principles and Standards for School Mathematics* (2000), and *Catalyzing Change in High School Mathematics* (2018) and is also a NAEP Mathematical Practice. The NCTM Process Standards include reasoning and proof, and states' standards for mathematical practice include constructing viable arguments; both are similar to the NAEP Mathematical Practice of Justifying and Proving. The NAEP Mathematical Practice of Abstracting and Generalizing is similar to a common state standard for mathematical practice about reasoning abstractly and quantitatively. Mathematical Modeling is in most states' standards for mathematical practice as well as a NAEP Mathematical Practice.

The NAEP Achievement Level Descriptions (ALDs; see Appendix A) provide examples of what students performing at the *NAEP Basic*, *NAEP Proficient*, and *NAEP Advanced* achievement levels should know and be able to do in terms of NAEP mathematics content and practices. Assessment developers need to create a pool of items that reflects the Framework and the range of achievement levels. Because consideration of achievement levels while developing items is important, some illustrations in Chapter 3 include ALD Notes for Item Developers. These notes provide descriptions of how NAEP achievement level language relevant to NAEP Mathematical Practices and content objectives is reflected in the given item and how the achievement level connection might be affected by revisions to the item.

Operationalizing the NAEP Mathematical Practices

A description of each NAEP Mathematical Practice follows. Although each practice is treated as distinct, they are interrelated with one another and with content, as is demonstrated in the examples provided throughout. In designing NAEP items, it may be impossible to completely isolate a particular mathematical practice in an item. When items assess multiple aspects of mathematics, it should be possible to identify a primary content focus and a primary practice focus. The former has been done on NAEP Mathematics Assessments for many years, and the latter should be possible moving forward. Further, the practices fundamentally intersect with, and develop in relation to, content. In this sense, the practices cut across grade levels, as well as across *NAEP Basic*, *NAEP Proficient*, and *NAEP Advanced* achievement levels. This approach to mathematical practices is reflected in policy and state standards, where mathematical content standards are offered and described by grade levels, while practices cut across grade levels. Just as some mathematics content objectives are more likely to interact with others in items, some mathematical practices are more likely to be found in connection with certain mathematics objectives. At the end of this chapter, Exhibits 3.25A–3.25C provide examples of where and how the five NAEP Mathematical Practices might be assessed within the NAEP mathematics content areas at each grade level. The tables are illustrative, not exhaustive, of ways practices could be assessed within content areas.

All released NAEP items used as exhibits in the Framework and in these specifications were accessed using the online NAEP Questions Tool (NCES, n.d.). Some examples are from other sources, including example items from the Smarter Balanced Assessment Consortium (SBAC), released items from the Partnership for Assessment of Readiness for College and Careers (PARCC), and adaptations of tasks from policy and curriculum documents. The source for each item is cited in related text description about the item.

NAEP Mathematical Practice 1: Representing

Representing: Recognizing, using, creating, interpreting, or translating among representations appropriate for the grade level and the mathematics being assessed.

Focus for Item Developers

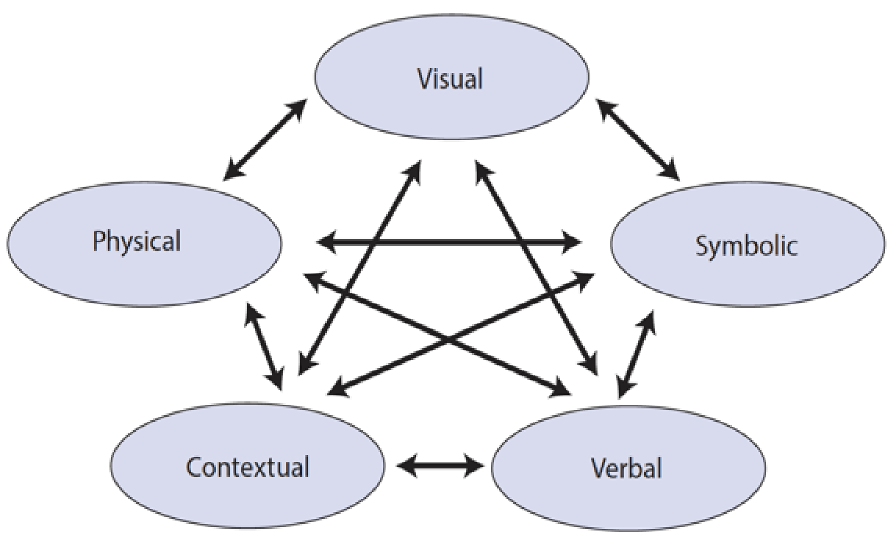
Examples of ways that students can engage in the NAEP Mathematical Practice of Representing include, but are not limited to,

- constructing visual representations of numbers, shapes, and data;
- translating from one mathematical representation to another;
- using representations as tools to solve problems; and
- building on, analyzing, and explaining representations created by others.

Each item associated with this practice should focus mathematical activity on representational actions within and across the modes of representation in Exhibit 3.2.

Representing mathematical ideas and using mathematical representations to make sense of and solve problems is central to mathematics. Students create representations themselves, or in collaboration with other students, and they reason from or translate between standard representations (e.g., graphs, tables, geometric drawings) (Lesh, Post, & Behr, 1987; NCTM, 2014). Tripathi (2008) argues that variety in representations "is like examining a concept through a variety of lenses, with each lens providing a different perspective that makes the picture (concept) richer and deeper" (p. 439). Exhibit 3.2, from *Principles to Actions* (NCTM, 2014, p. 25) illustrates some of the types of representation and the relationships among them.

Exhibit 3.2. Types and Connections Among Mathematical Representations

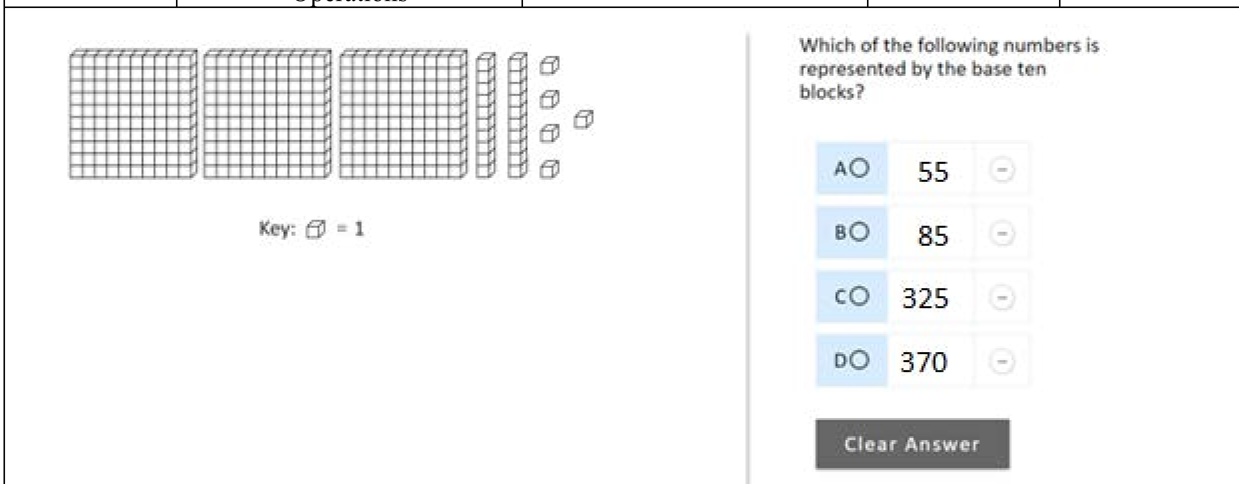


According to the National Research Council (NRC, 2009), students, especially young ones, benefit from using physical objects or acting out processes during problem solving. Base 10 blocks (or blocks/tiles representing other bases), fraction strips/bars, red–black integer tiles, and algebra tiles are all examples of physical representations of number and operation that are used to enhance students' understanding of concepts in elementary and middle grades. These visual and physical representations connect, eventually, to symbolic representations as well. Visual representations also play a particularly powerful role in helping students make sense of problems and understand mathematical concepts and procedures. For instance, arrays of squares in a grid can be used to represent area models for mathematical operations such as multiplication and division in early elementary grades, then later for multiplication of algebraic expressions. Additionally, students create, use, and reason about multiple representations for a given mathematical idea or relationship in contextually relevant ways.

The grade 4 item in Illustration 3.1 is adapted from Exhibit 3.3 in the Framework. The item provides an image of base 10 blocks and asks students to determine the number shown. In answering the question, students connect a visual representation of a number to its symbolic representation in base 10. The item is framed to elicit a basic level response.

Illustration 3.1. **Representing Example:** **Base 10 Blocks**
adapted from **Exhibit 3.3**

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
4	Number Properties and Operations	Representing	Num – 1.b	SR – MC



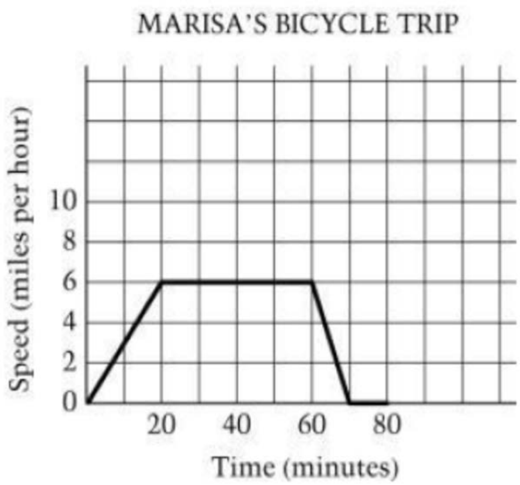
Scoring Information	
Key	C. 325
ALD Notes for Item Developers	
Basic	The item assesses understanding of a visual representation of a familiar set, whole numbers.
Proficient	The item could be revised to present a representation of a whole number using base 10 blocks and ask for a different representation of the same number also using base 10 blocks.
Advanced	The item could be revised to require a description of why a base 10 representation of a number is incorrect along with a corrected representation.

The item in this illustration is adapted from a NAEP item. The original version of this item appeared in the 2017 grade 4 NAEP Mathematics Assessment with NAEP Item ID 2017-4M9 #15 M347601.

The grade 8 item in Exhibit 3.4, from the 2003 NAEP Mathematics Assessment, demonstrates how students might provide a verbal representation from a graphical representation, or generate several alternative representations based on a problem situation. The item asks a student to take a graphical representation and work backward to a context that could fit that representation.

Alternatively, students could be asked to create their own graphical representation of a bicycle trip over time from a given verbal description of a trip. More realistic graphs of trips could be presented; for example, the item might offer a graph of a bicycle trip with more of a range and variety of speeds, including where the speed is zero at times mid-trip. Students could be given several different explanations that were provided by hypothetical students and asked to decide if those explanations correctly match the representation in the graph, or what an alternative explanation might be.

Exhibit 3.4. Grade 8 (and/or Grade 12) NAEP *Bicycle Trip* Item



The graph above represents Marisa's riding speed throughout her 80-minute bicycle trip. Use the information in the graph to describe what could have happened on the trip, including her speed throughout the trip.

During the first 20 minutes, Marisa

From 20 minutes to 60 minutes Marisa

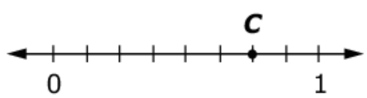
From 60 minutes to 80 minutes Marisa

Illustration 3.2 is adapted from Exhibit 3.5 in the Framework. The Smarter Balanced (SBAC) item shown provides a point on a number line that represents a distance, along with additional written information. As they work to solve the problem, students are expected to engage with the measurement represented on the number line in conjunction with some additional information, recognize the representation of a fraction, and apply it within the given context.

Illustration 3.2. Representing Example: Number Line
adapted from **Exhibit 3.5**

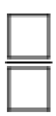
Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
4	Number Properties and Operations	Representing	Num – 3.f	SCR – FIB

Valeria and Diego walked home from school. The distance Valeria walked, in miles, is represented by point *C* on the number line.



Diego walked $\frac{2}{8}$ mile less than Valeria.

Enter the distance Diego walked as a fraction of a mile.



Scoring Information	
Key	4/8 (or equivalent)
ALD Notes for Item Developers	
Basic	The item could be revised to remove reference to Diego and ask for the fraction of a mile that Valeria walked, as represented by point *C* on the number line.
Proficient	The item assesses use of a number line to solve a fraction computation problem involving comparing distances.
Advanced	The item could be revised to ask for the selection of a response that describes how the number line could be used to determine the distance Diego walked.

The item in this illustration is adapted from an SBAC item with Item ID 3218, aligned to CCSS-M objective 5.NF.A.2.

Illustration 3.3 is based on Exhibit 3.6 in the Framework. Similar to the previous item, the SBAC item shown asks students about two more ways of representing. In it, students select the written statement that could be represented by the given equation, connecting a context to a symbolic representation.

Illustration 3.3. Representing Example: Connecting Context to a Symbolic Representation *based on* Exhibit 3.6

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
8	Number Properties and Operations	Representing	Num – 3.e	SR – MC

Which situation can be represented by this equation?

$4 \div \frac{1}{8} = \square$

Ⓐ Jack has 4 pieces of fabric. Each piece is $\frac{1}{8}$ of a yard long. How many yards of fabric does Jack have?

Ⓑ Jack has 4 pieces of fabric. He gets $\frac{1}{8}$ more yards of fabric. How many yards of fabric does Jack have now?

Ⓒ Jack has 4 yards of fabric. He gives away $\frac{1}{8}$ of his pieces of fabric. How many pieces of fabric does Jack have left?

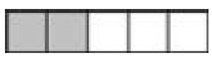
Ⓓ Jack has 4 yards of fabric. He cuts the fabric into pieces $\frac{1}{8}$ of a yard long. How many pieces of fabric does Jack have?

Scoring Information	
Key	D. Jack has 4 yards of fabric. He cuts the fabric into pieces 1/8 of a yard long. How many pieces of fabric does Jack have?
ALD Notes for Item Developers	
Basic	The item assesses the translation from one representation of a fraction operation (numeric) to another (verbal).
Proficient	The item could be revised to provide a number line from 0 to 4 partitioned into eights and ask for an explanation for how the number line represents the quotient of 4 and 1/8.
Advanced	The item could be revised to provide the expression 4 ÷ (1/8) and a correct visual measurement representation of the quotient, but an incorrect numerical representation of the quotient. The directive could be to explain the relationship between the visual and numeric representations of the quotient provided and determine whether each could be a correct representation.

The item in this illustration is based on an SBAC item with Item ID 3274, aligned to CCSS-M objective 5.NF.B.7b.

Translating from one mathematical representation to another is a component of the practice of representing. For instance, the grade 4 item on the left in Illustration 3.4 (example) asks students to write a fraction to describe the shaded part of a figure, a translation from the visual to the numeric. In contrast, the grade 8 item on the right in Illustration 3.4 (nonexample) asks students to choose the set of fractions ordered from least to greatest. Although a response to the nonexample item reveals something about what a student knows about fractions, it does not assess the NAEP Mathematical Practice of Representing because the selection of the correctly ordered list does not meaningfully convey understanding of the representing of relative fraction size.

Illustration 3.4. Example and Nonexample of Representing

<table>
<tr><th>Grade Level</th><th>Content Area</th><th>Assessed Practice(s)</th><th>Objective ID</th><th>Item Format</th></tr>
<tr><td>left: 4</td><td rowspan="2">Number Properties and Operations</td><td>Representing</td><td>Num – 1.e</td><td>SCR – FIB</td></tr>
<tr><td>right: 8</td><td>Other</td><td>Num – 1.i</td><td>SR – MC</td></tr>
<tr><td colspan="2">Example

What fraction of the figure is shaded?
Answer: ________________</td><td colspan="3">Nonexample
In which of the following are the three fractions arranged from least to greatest?
A. $\frac{2}{7}, \frac{1}{2}, \frac{5}{9}$
B. $\frac{1}{2}, \frac{2}{7}, \frac{5}{9}$
C. $\frac{1}{2}, \frac{5}{9}, \frac{2}{7}$
D. $\frac{5}{9}, \frac{1}{2}, \frac{2}{7}$
E. $\frac{5}{9}, \frac{2}{7}, \frac{1}{2}$</td></tr>
<tr><td colspan="5">Scoring Information</td></tr>
<tr><td>Key</td><td>2/5</td><td>Key</td><td colspan="2">2/7, 1/2, 5/9</td></tr>
</table>

The grade 4 item in this illustration is based on a NAEP item. The original version of this item appeared in the 2007 NAEP Mathematics Assessment with NAEP Item ID 2007-4M7 #6 M139301.
The grade 8 item in this illustration is based on a NAEP item. The original version of this item appeared in the 2007 NAEP Mathematics Assessment with NAEP Item ID 2007-8M9 #12 M013631.

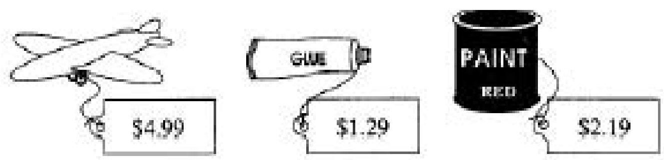
The shaded figure in the example item in Illustration 3.4 is central to the item's assessment of representing. However, the inclusion of an image in an item does not automatically address the NAEP Mathematical Practice of Representing.

Items may include images to convey information that could be provided another way, such as through written text. For a given image to be associated with the practice of representing, the image would need to be a representation with which students engage mathematically and that is critical to the solution process.

Though the item in Illustration 3.5 uses an image to convey information critical to solving the problem (each cost is needed), the image itself is not essential for the mathematical activity required, nor is there a need for students to construct or analyze representations to solve the problem. Therefore, the item in Illustration 3.5 does not assess the NAEP Mathematical Practice of Representing.

Illustration 3.5. Representing Nonexample: Image Does Not Address Representing

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
4	Number Properties and Operations	Other	Num – 3.f	SR – MC



Chen bought one model plane, one tube of glue, and one can of paint. The cost of each item is shown in the figure above. There was no sales tax. How much change should he have gotten back from $10?

A. $1.50

B. $1.53

C. $1.63

D. $1.73

Scoring Information	
Key	B. $1.53

The item in this illustration is based on a NAEP item. The original version of this item appeared in the 1990 grade 4 NAEP Mathematics Assessment with NAEP Item ID 1990-4M9 #7 M026131.

NAEP Mathematical Practice 2: Abstracting and Generalizing

> *Abstracting and Generalizing: Decontextualizing, identifying commonality across cases, items, problems, or representations, and extending one's reasoning to a broader domain appropriate for the grade level and the mathematics being assessed.*

Focus for Item Developers

Engaging in the NAEP Mathematical Practice of Abstracting and Generalizing involves one or both of the processes of abstracting or generalizing.

- *Abstracting* is the process of decontextualizing ideas in a given problem or context and expressing, representing, and manipulating them in a manner independent of initial contextual references (Scheiner & Pinto, 2016).
- *Generalizing* is the process or outcome of at least one of the following actions (Ellis, 2011, p. 311):
  - identifying commonality across cases,
  - extending one's reasoning beyond the range in which it originated, or
  - deriving broader results from particular cases.

Abstracting

Students learning and doing mathematics also engage in the practice of abstracting and generalizing. An essential element of mathematical learning and problem solving is the ability to reason abstractly and to develop, test, and refine generalizations. In reasoning abstractly, students engage in the process of *decontextualizing*: Students abstract ideas in a given problem or context and express and manipulate them in a manner independent of their contextual references. Decontextualizing can foster an understanding of the relationships among problem contexts and written or symbolic forms, as well as an understanding of how mathematical expressions might be transformed to facilitate a solution strategy. Abstracting is also a critical activity for fostering generalizing; it enables a consideration of concepts and relationships decontextualized from specific examples or cases, which can support the formation of a more general rule or relationship.

Young students, for instance, can notice patterns of additive commutativity, such as $3 + 7$ yielding the same sum as $7 + 3$. In this instance, decontextualization would include finding a way to represent this relation independent of particular numbers, as a more general identity. Younger students might express this general identity verbally or with pictures, or with the use of a generic example. Older students might express this identity algebraically as $a + b = b + a$. Reasoning abstractly can also support recognizing similar mathematical structures across different problems or domains. For example, one could see the multiplication of two binomials $(2x + 7)(3x + 2)$ as a more general version of multiplying 27 by 32.

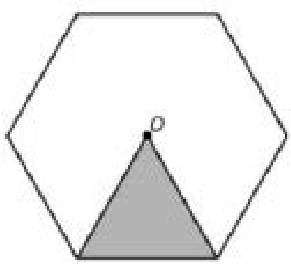
Consider the grade 8 Geometry item in Exhibit 3.7, from the 2017 NAEP Mathematics Assessment. This item requires students to express the area of the hexagon in terms of the area of the given shaded triangle. Students are then asked to extend their reasoning to a 10-sided figure. Thus, students are first challenged to reason structurally by mentally comparing the area of the triangle formed by the hexagon's center and two adjacent vertices with the area of the entire figure. Students are then further tasked with extending their reasoning from the specific case of the hexagon to another regular polygon.

Although a student could solve the problem in Exhibit 3.7 by drawing a 10-sided polygon and the specified triangle, and then counting the number of triangles that comprise the polygon, a student could also carry out this operation mentally rather than drawing it out. Also, the item could be revised to elicit decontextualizing beyond the hexagon, thinking about the relationship between the specified triangle and any regular polygon. In the later grades, students could be expected to express their reasoning algebraically and develop and prove a conjecture about the general relationship between the triangle and any *n*-sided regular polygon.

Exhibit 3.7. Grade 8 NAEP Geometry Item

Point *O* is the center of the regular hexagon shown.

The shaded triangle is formed by *O* and two adjacent vertices of the hexagon and has an area of *T*.



What is the area of the hexagon in terms of *T* ?

Area=

Point *P* is the center of a regular polygon with 10 sides.

A triangle is formed by *P* and two adjacent vertices of the polygon and has an area of *V*.

What is the area of the polygon in terms of *V* ?

Area=

Abstracting can occur across different domains. It can be addressed in reasoning about figures and their relationships in geometry, about number theory in number properties and operations, or about equivalence or functional relationships in algebra. How one decontextualizes or reasons with structure will differ across the domains, but these are processes students can employ in all five content areas included in the NAEP Mathematics Assessment.

Generalizing

Mathematics education researchers and policymakers have defined generalizing in a number of ways. Historically, generalization has been defined as an individual, cognitive construct (e.g., Carraher, Martinez, & Schliemann, 2008), where generalization is the act of identifying a property that holds for a larger set of mathematical objects or conditions than the number of individually verified cases. For instance, Harel and Tall (1991) described generalization as the process of "applying a given argument in a broader context" (p. 38), and Radford (2007) argued that generalization involves identifying a commonality based on particulars and then extending it to all terms.

More recently, researchers have begun to address generalizing as a construct that is both social and cognitive; that is, it can occur either individually or collectively. Therefore, for NAEP, generalizing is an individual or collective practice of (a) identifying commonality across cases, (b) extending reasoning beyond the domain in which it originated, and/or (c) deriving broader results from particular cases (Ellis, 2007). Its social dimensions make it relevant to the NAEP Collaborative Mathematics practice.

Several aspects of mathematical reasoning can foster generalizing. As previously mentioned, abstracting and decontextualizing are important mental actions that support generalizing. Other actions that support generalizing include visualizing, focusing, reflecting, connecting, and expressing. *Visualizing* involves seeing patterns or structural relationships, as well as imagining a set of relationships beyond what is perceptually available. *Focusing* is attending to particular details, characteristics, properties, or relationships above others. This can include examining a particular case in a pattern or attending to figural or numerical cues. *Reflecting* involves actions such as thinking back on the operations one has carried out, observing one's method in solving problems, or examining the rules that govern a given pattern. *Connecting* is the identification of relationships among tasks, representations, or properties. Making connections between representations or identifying and operating on structural similarities can foster the development of generalizations. Finally, *expressing* involves depicting a generalization verbally or in writing. Describing generalizations in words can support the subsequent development of algebraically represented generalizations. When expressing involves both representing and generalizing, more than one NAEP Mathematical Practice could be engaged. In these cases, the practice emphasized to a greater extent in the item should be selected as the primary practice.

Like abstracting, generalizing can occur across the content areas and grade bands. Existing NAEP Mathematics Assessment items contain a number of generalization tasks in which students are asked to determine a rule guiding the pattern of number terms in a sequence. In some items, potential rules are provided for students who are prompted only to attend to the action required to move from one term in the sequence to the next. In other items, students must determine a rule themselves, such as for the grade 12 item in Exhibit 3.8. The item in Illustration 3.6 is a *NAEP Advanced* variant of the item in Exhibit 3.8. The item shown removes the scaffolding of parts (a) and (b) from the original item. It is worth noting that for items such as the one in Exhibit 3.8, there could be any number of non-equivalent rules to describe the pattern, so it may be more appropriate to ask students to provide "a" rule rather than "the" rule (e.g., as in Illustration 3.6).

Notice that for part c of the grade 12 item in Exhibit 3.8 and the adaptation in Illustration 3.6, students are expected to write a formal algebraic rule for moving from the n^{th} term to the $(n + 1)^{\text{st}}$ term of Sequence I by identifying an explicit rule for the n^{th} term of Sequence II. In other items, students may be tasked with determining a recursive, rather than explicit, rule to find the n^{th} term in a sequence.

Exhibit 3.8. Grade 12 NAEP Number Pattern Item

Sequence I: 3, 5, 9, 17, 33, . . .

Sequence I, shown above, is an increasing sequence. Each term in the sequence is greater than the previous term.

a. Make a list of numbers that consists of the positive differences between each pair of adjacent terms in Sequence I. Label the list Sequence II.

b. If this same pattern of differences continues for the terms in Sequence I, what are the next two terms after 33 in Sequence I ?

6th term ______________________

7th term ______________________

c. Write an algebraic expression (rule) that can be used to determine the n^{th} term of Sequence II, which is the difference between the $(n+1)^{\text{st}}$ term and the n^{th} term of Sequence I.

Illustration 3.6. Abstracting and Generalizing Example: Write a Rule to Describe a Pattern *adapted from* **Exhibit 3.8**

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
12	Algebra	Abstracting and Generalizing	Alg – 1.b	SCR

The sequence shown is increasing. Each term in the sequence is greater than the previous term.

3, 5, 9, 17, 33, . . .

The same pattern continues for the terms in the sequence. Write an algebraic expression (rule) to represent the difference between the $(n + 1)^{\text{st}}$ term and the n^{th} term of the sequence.

Scoring Information	
Key	2^n
ALD Notes for Item Developers	
Basic	The item could be revised to assess the identification of the pattern of differences (i.e., part [a] in Exhibit 3.8).
Proficient	The item could be revised to assess the extension of the pattern of differences (i.e., parts [a] and [b] in Exhibit 3.8).
Advanced	The item assesses use of structures and patterns to determine a complex rule.

The item in this illustration is based on a NAEP item. The original version of this item appeared in the 2005 grade 12 NAEP Mathematics Assessment with NAEP Item ID 2005-12M3 #17 M095401.

Determining a rule for a pattern is a common focus of grade 4 generalization items, such as the adapted 2011 grade 4 TIMSS item shown in Illustration 3.7 (International Association for the Evaluation of Educational Achievement, 2013). The original TIMSS item was multiple choice, with response options providing choices for Steve's rule. Here it has been modified to a fill-in-the-blank item requiring students to determine correct values for the rule, avoiding the opportunity for students to check provided response options to determine which could be the rule.

Illustration 3.7. Abstracting and Generalizing Example: Complete a Rule to Describe a Pattern

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
4	Algebra	Abstracting and Generalizing	Alg – 1.a	SCR – FIB

Steve used the same rule to get the number in the □ from the number in the △.

Complete this rule so that it could be Steve's Rule.

multiply by ☐, then add ☐

Scoring Information

Key	multiply by 2, then add 2

The item in this illustration is adapted from a TIMSS item. The original version of this item appeared in the 2011 grade 4 TIMSS assessment with Item ID M031251.

Students can also be challenged to engage in the processes of generalizing in items that do not rely on pattern sequences, as in Exhibit 3.9. This item could support a number of possible generalizing processes, as well as the opportunity for abstracting. For instance, one could consider that for each coin (nickel, dime, quarter), there are two possible outcomes, H or T. Thus, a student could either systematically list outcomes to determine that there are 8 total outcomes or begin to think structurally to reason that for three coins and two outcomes per coin, there must be $2^3 = 8$ total outcomes. Alternatively, through systematic listing, a student could determine that there are 1 + 3 + 3 + 1 outcomes, corresponding to 1 outcome with exactly zero Ts, 3 outcomes with exactly one T, 3 outcomes with exactly two Ts, and 1 outcome with exactly three Ts. Extending to the 4-coin case, for instance, students might determine that the number of outcomes is 1 + 4 + 6 + 4 + 1, corresponding to 1 outcome with exactly zero Ts, 4 outcomes with exactly 1 T, 6 outcomes with exactly 2 Ts, 4 outcomes with exactly three Ts, and 1 outcome with exactly four Ts (and symmetrically but opposite for the number of Hs).

Exhibit 3.9. Grade 8 and/or Grade 12 Task (Adapted from a Grade 8 NAEP Item)

Three students each have a coin, one has a nickel, one has a dime, and the third student has a quarter. They flip their coins at the same time. Each coin can land either heads up (H) or tails up (T). List all the different possible outcomes for how the coins could land in the chart below. The list has been started for you.

Nickel	Dime	Quarter
H	H	H
H	H	T

What if a 4th student joins the group with a half-dollar coin? How many different ways could the 4 coins land? What if a 5th student joined with a penny–how many different ways could the 5 coins land?

One aspect of generalizing is identifying commonality across cases. Students might notice that the outcomes for the 3-coin and 4-coin cases can be structured according to the rows in Pascal's triangle. Or, students might reason that, like the 3-coin case, each of the positions in the 4-coin case has two possible outcomes, H or T, and thus the total number of possible outcomes must be $2^4 = 16$, and, more generally, for n coins, 2^n. An item like the one in Exhibit 3.9 affords a number of rich generalizing opportunities, regardless of whether students are expected to recognize that 2^n is the sum of the coefficients of the binomial expression $(a + b)^n$ (e.g., $2^4 = 1 + 4 + 6 + 4 + 1$).

An item assessing generalizing may call on structural reasoning, breaking mathematical components of an item apart to identify the building blocks needed to answer a question (Cuoco, Goldenberg, & Mark, 1996; Küchemann & Hoyles, 2009). Thus, a distinction needs to be made between items that ask students to reason structurally and items that prompt students to identify or apply known quantities or properties. Consider the grade 8 item in Illustration 3.8. For each question a set of geometric objects is given and a single example is sufficient to determine the correct response. Students do not need to consider the structure of each set. Therefore, this item does not ask students to reason structurally about described sets of geometric objects and is not an assessment of the NAEP Mathematical Practice of Abstracting and Generalizing (see Chapter 4 for additional information about the grid item type represented in Illustration 3.8).

Illustration 3.8. Abstracting and Generalizing Nonexample: Recognizing Properties of Geometric Objects

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
8	Geometry	Other	Geom – 3.g	SR – grid

Three questions about lines and angles in the plane are given in the table.

Indicate whether the answer to each question is 0, 1, 2, 4, or 8.

Make one selection for each question to show your answer.

Questions	0	1	2	4	8
At how many points do two different parallel lines meet?	○	○	○	○	○
How many right angles are formed by a pair of perpendicular lines?	○	○	○	○	○
How many angles that measure less than 180 degrees are formed when two parallel lines are cut by a transversal?	○	○	○	○	○

Clear Answer

Scoring Information

Key

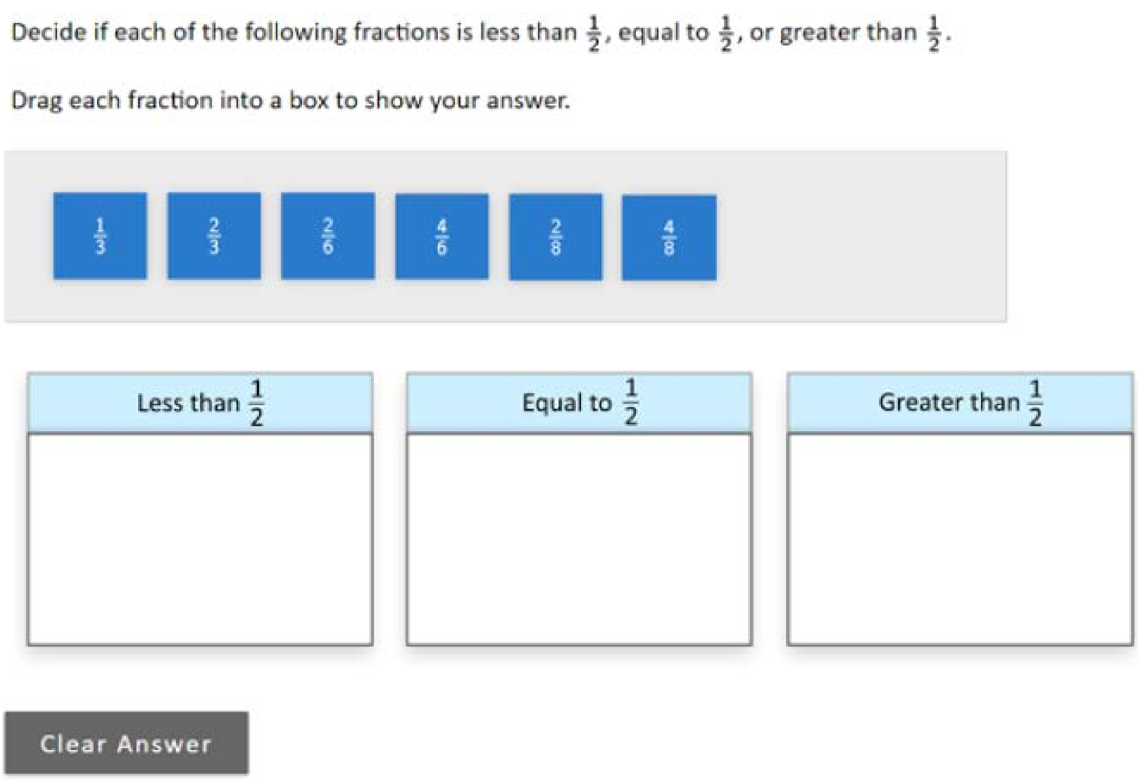
Questions	0	1	2	4	8
At how many points do two different parallel lines meet?	◉	○	○	○	○
How many right angles are formed by a pair of perpendicular lines?	○	○	○	◉	○
How many angles that measure less than 180 degrees are formed when two parallel lines are cut by a transversal?	○	○	○	○	◉

The item in this illustration is based on a NAEP item. The original version of this item appeared in the 2017 grade 8 NAEP Mathematics Assessment with NAEP Item ID 2017-8M3 #10 M3821MS.

Since the NAEP Mathematical Practice of Abstracting and Generalizing involves reasoning about mathematical structures and systems, items that focus on concrete examples likely do not assess this practice. Consider the grade 4 item in Illustration 3.9. In this item, students are asked to compare given fractions to the benchmark number 1/2. Note that they are not asked to determine the structure of a fraction that is less than, equal to, or greater than 1/2. The thought processes behind a student's matching results are unknown. Instead, the evidence provided from a response to this item indicates whether or not the fractions were compared correctly (see Chapter 4 for additional information about the matching item type in Illustration 3.9).

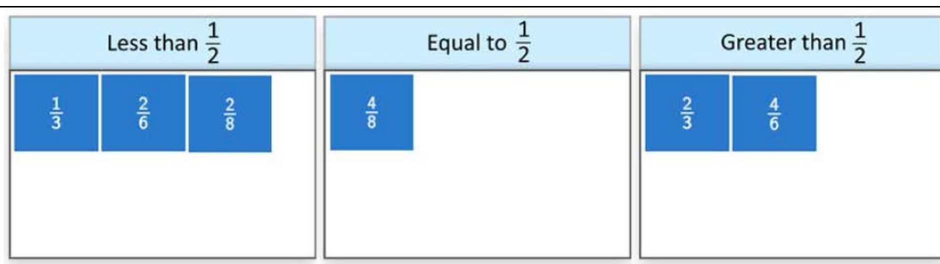
Illustration 3.9. Abstracting and Generalizing Nonexample: Using Benchmarks

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
4	Number Properties and Operations	Other	Num – 1.i	SR – matching



Scoring Information

Key



The item in this illustration is based on a NAEP item. The original version of this item appeared in the 2017 grade 4 NAEP Mathematics Assessment with NAEP Item ID 2017-4M1 #6 M3714MS.

NAEP Mathematical Practice 3: Justifying and Proving

Justifying and Proving: Creating, evaluating, showing, or refuting mathematical claims in developmentally and mathematically appropriate ways.

Focus for Item Developers

Explaining why something is true or not true is an important aspect of mathematical argumentation. However, not all mathematical arguments involve the NAEP Mathematical Practice of Justifying and Proving.

While the practice includes creating, evaluating, showing, or refuting mathematical claims, distinctions to note for the NAEP Mathematical Practice of Justifying and Proving include:

- *Justifying* involves a deductive argument demonstrating why a statement or claim is true or not true generally (i.e., for all cases).
- *Proving* involves the formal presentation of a valid justification.
- Examples alone do not suffice as a mathematical justification or proof except for proofs by exhaustion or counterexample.
- Explanation of mathematical reasoning aligns with the practice only when the item requires justifying a general statement, not specific instance(s) (e.g., Illustrations 3.12a, 3.14, and 3.15).
- Items to assess generalizing and justifying differ.
    - Given information, a student generalizes (e.g., Illustration 3.6).
    - Given a general statement as a claim, a student justifies it (e.g., Illustration 3.11).

Justifying and proving are essential in all content areas and grade levels. Traditionally, proof was viewed as a form of mathematical argumentation pertaining first to high school geometry and not visited again until pre-calculus courses with proofs of trigonometric identities and proofs by mathematical induction. However, this changed in the last quarter of the 20th century. The *Principles and Standards for School Mathematics* emphasized the importance of justifying and proving at all levels of mathematics, noting that "reasoning and proof should be a consistent part of students' mathematical experience in prekindergarten through grade 12" (NCTM, 2000, p. 56). Similarly, state standards highlight the activities students engage in as they learn to create valid mathematical arguments: making and investigating conjectures, developing particular forms of argument (e.g., deductive), and using a variety of proof methods (e.g., direct, counterexample). These are all considered components of the practice of justifying and proving.

Mathematical justification includes creating arguments, explaining why conjectures must be true or demonstrating that they are false, exploring special cases or searching for counterexamples, understanding the role of definitions and counterexamples, and evaluating arguments (Ellis, Bieda, & Knuth, 2012). A valid justification should show why a statement or conjecture is true or not true generally (i.e., for all cases) and, especially by grades 8 and 12, should do so by providing a logical sequence of statements, each building on already established statements, ideas, or relationships.

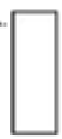
A justification is *not* based on authority, perception, popular consensus, or examples alone. As students engage in justifying, they may be tempted to rely on external sources to verify their ideas, such as their teacher or a textbook (Harel & Sowder, 1998). Students may also want to use examples to support their claims, concluding that a conjecture must be true because it holds for several different cases. Examples can and do play an important role in justifying and proving, particularly in terms of helping students make sense of statements, gain a sense of conviction, or revealing an underlying structure that could lead to a proof. But they do not suffice as a mathematical justification or proof except for proofs by exhaustion or counterexample.

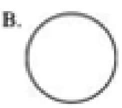
Consider the item in Illustration 3.10, which asks students to choose a counterexample to Alan's claim from a set of response options. The required action reflects emerging development of the practice of justifying and proving.

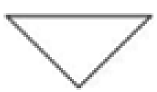
Illustration 3.10. Justifying and Proving Example: Choose the Counterexample

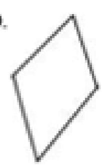
Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
4	Geometry	Justifying and Proving	Geom – 1.e	SR – MC

Alan says that if a figure has four sides, it must be a rectangle. Gina does not agree. Which of the following figures shoes that Gina is correct?

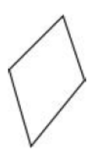
A.


B.


C.


D.


Scoring Information

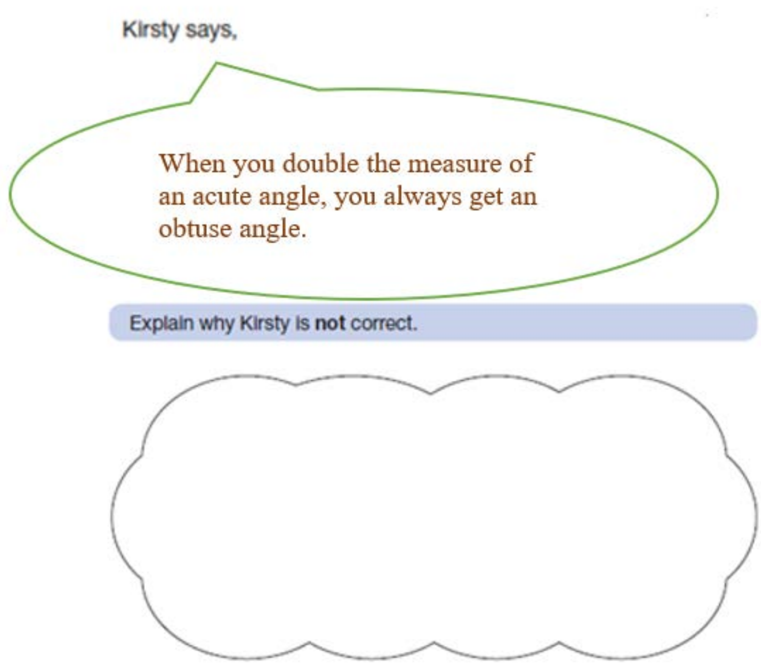
Key	D. 

The item in this illustration is based on a NAEP item. The original version of this item appeared in the 2003 grade 4 NAEP Mathematics Assessment with NAEP Item ID 2003-4M6 #7 M046401.

The adapted *Key Stage* released item in Illustration 3.11 provides a mathematical statement and asks for an explanation of why the statement is incorrect. As with the previous illustration, this item, at a minimum, requires presentation of a counterexample, a single example of when doubling an angle measure results in an angle that is not obtuse. As posed, the item primarily assesses the practice of justifying, even though a student might respond in a way that generalizes the range of angle sizes for which the given statement is false. Since that type of response is not requested, the item does not primarily assess the practice of generalizing.

Illustration 3.11. Justifying and Proving Example/Abstracting and Generalizing Nonexample: Generate a Counterexample

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
4	Geometry	Justifying and Proving	Geom – 1.b	SCR



Scoring Information

Key

An explanation that includes a correct counter example, e.g. • When you double 10° it is not obtuse • 2 × 27° = 54° • Double 45° is a right angle not obtuse **OR** An explanation that demonstrates where the statement in the question is not correct, e.g. • If the acute angle is less than 45° then doubling it will be less than 90°, so it won't be obtuse (more than 90°).	**Do not** accept vague or incomplete explanations, e.g. • Sometimes it will be acute • Some acute angles are half an obtuse angle, but not all • When you double an acute angle, you get a right angle **Do not** accept explanations which include incorrect mathematics or incorrect information that is relevant to the explanation, e.g. • 20°C × 2 = 40°C • 20% x 2 = 40%

The item in this illustration is adapted from an England *Key Stage 2* item. The original version of this item appeared as Item 13 in the 2019 administration of *Paper 3: Reasoning*.

Definitions are often used to justify mathematical statements. However, applying a definition is not necessarily engaging in the NAEP practice of Justifying and Proving. The grade 12 item in Illustration 3.12a, adapted from a 2009 NAEP item, provides a mathematical claim, "y is a function of x," about the particular values in the given table. In generating a correct response to the item, students are likely to use some form of the definition of function to write the requested explanation. However, the item asks about the particular case of the x-y relation in the table and does not call on students to create an explanation that is generally true (e.g., for a class of cases). As in the sample explanation in the key, students might make a general explanation, but the item does not require it. Therefore, the item does not assess the NAEP Mathematical Practice of Justifying and Proving.

Illustration 3.12a. Justifying and Proving Nonexample: Supporting with a Definition

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
12	Algebra	Other	Alg – 1.g	SCR

This table shows all of the ordered pairs (x, y) that define a relation between the variables x and y.

x	-2	-1	0	1	2	3
y	3	0	-1	0	3	8

Explain whether or not y is a function of x.

Scoring Information

Key	Sample Explanation: y is a function of x because for each x-value (domain) there is only one y-value (range) that is associated with it.

The item in this illustration is adapted from a NAEP item. The original version of this item appeared in the 2009 grade 12 NAEP Mathematics Assessment with NAEP Item ID 2009-12M2 #7 M1906E1.

By contrast, consider the variant of Illustration 3.12a shown in Illustration 3.12b. The item in Illustration 3.12b calls for a justification that is generally true for a class of examples, in this case all linear relationships between x and y.

Similarly, Illustrations 3.12c and 3.12d, for grade 8 and grade 4, respectively, show items where students are called on to determine and explain the validity of general mathematical relationships. In Illustrations 3.12b, 3.12c, and 3.12d, students might leverage examples to demonstrate why a statement is generally true, yet the items require justifying general statements rather than specific instances.

Illustration 3.12b. Grade 12 Justifying and Proving Example: Supporting with a Definition

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
12	Algebra	Justifying and Proving	Alg – 1.g	SCR

The equation $y = cx + \mathrm{d}$, where c and d are constants and $c \neq 0$, defines a relationship between the variables x and y. Explain whether or not y is a function of x.

Sample Student Responses

(1) If y is a function of x, then for each x there is exactly one y. So what if it is not a function? Let's pick two values for y and say they correspond to the same x. We'll call these values of y, w and z. Since w and z correspond to the same x, $w = cx + d$ and $z = cx + d \rightarrow w = z$. Since w and z are the same number, there are not two values of y that correspond to the same x. Therefore, y is a function of x.

(2) y is a function of x because for each x, when you multiply by whatever c is you get exactly one number and then you add whatever d is and still get one output. So, there is one output y for one input x. The definition of function is that for every input there is only one output, so y is a function of x.

Illustration 3.12c. Grade 8 Justifying and Proving Example: Supporting with a Definition

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
8	Number Properties and Operations	Justifying and Proving	Num – 1.b	SCR

Ella makes this statement.

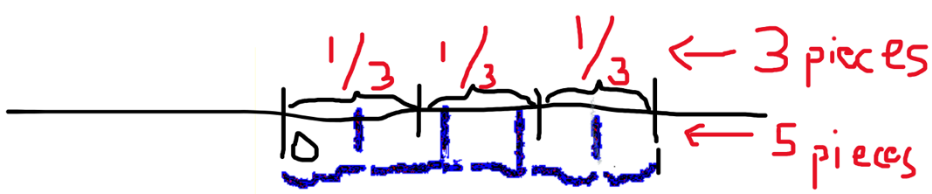
The opposite of the opposite of a number is the number.

Use a number line to explain whether or not Ella's statement is true.

Sample Student Response

(1) [student uses additive inverse]
Ella's statement is true. For example, the opposite of 3 is –3, since the opposite of a number and the number are on the opposite side of 0 on the number line but the same distance from 0. Using the same thinking, the opposite of –3 is 3. Therefore, the opposite of the opposite of 3 is 3.
This reasoning can be applied to any number. Let's say the number is n. The opposite of n is $-n$. The opposite of $-n$ is the same distance from 0 on the number line but on the opposite side from $-n$, so it is n. Therefore, the opposite of the opposite of n is n.

(2) [student uses multiplicative inverse]
If the opposite of a number is 1/number, then flipping it again would give the number back. This is because, on a number line, for example, 1 over 3 means the fraction that goes into one, three times, like in the picture. There are 3 pieces of 1/3. And 1 over 5 means the fraction that goes into one five times, three are 5 parts and each is 1/5, etc., so 1/number would go into 1 that number of times.



The opposite of 1/number would be asking how many times does 1 go into number. Well the answer is always the number – for example, 1 goes into 3, 3 times, 1 goes into 5, five times, and 1 goes into number, number times. So, the opposite of number is 1/number and the opposite of 1/number is number so the opposite of the opposite of a number is number and Ella is saying a true thing.

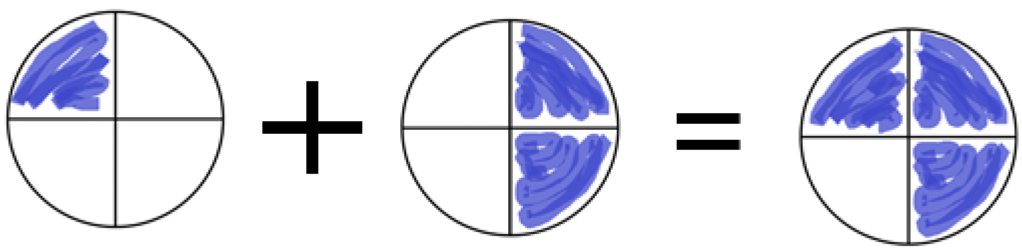
Illustration 3.12d. Grade 4 Justifying and Proving Example: Supporting with a Definition

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
4	Number Properties and Operations	Justifying and Proving	Num – 3.a	SCR

Terry is adding two fractions with the same denominator. In his answer, Terry adds the numerators but keeps the denominator the same. Explain whether or not Terry's answer is correct.

Sample Student Response

Terry's answer is correct because fractions are out of a whole.
Here is an example:



This picture shows that 1/4 + 2/4 = 3/4. This works because the wholes are the same size and the parts are the same size. When you combine the parts, the whole doesn't change. You just add the parts. You could do the same thing, no matter how many parts, as long as each whole was the same number of parts.

A proof can have many different forms, including narrative, pictorial, diagram, two-column, or algebraic forms. The form used to represent a mathematical proof is valid as long as it communicates the proof's essential features, namely, that it contains logically connected mathematical statements that are based on valid definitions and theorems. For instance, consider the grade 4 item in Exhibit 3.10. Some students may use specific examples in their arguments, but a complete response to this item requires students to indicate why the claim is true for all numbers.

Exhibit 3.10. Grade 4 Number Properties and Operations Proof Item

Elise claims that if you multiply any whole number by 6, you will always get an even number for the answer. Provide an argument for why Elise is correct.

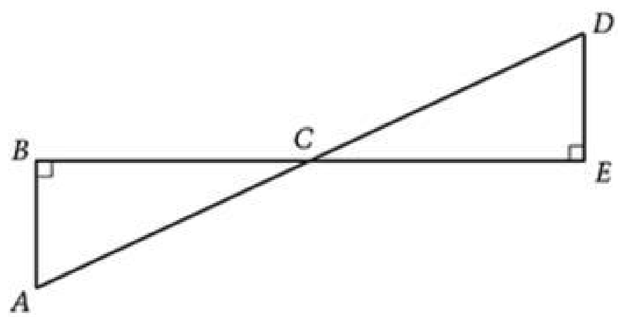
A grade 4 proof for the claim in Exhibit 3.10 could involve demonstrating with either pictures or symbols that the answer can always be separated into two equal parts, because 2 is a factor of 6, or that the answer can always be divided by 2 or cut in half because 2 already divides 6. An argument such as 6 x NUMBER = 3 x NUMBER + 3 x NUMBER might also be provided by fourth graders, demonstrating symbolically that the result can be split into two equal parts.

Arguing from examples alone is not a justification, but in providing examples students may discover the key piece to demonstrate that 2 will always be a factor of the product.

A formal proof is a specific type of argument "consisting of logically rigorous deductions of conclusions from hypotheses" (NCTM, 2000, p. 55). In grade 12, students are expected to develop formal mathematical proofs. A proof uses definitions and theorems that are available without further justification, and a proof is valid only if the assumptions upon which it relies have already been shown to be true.

Often, the phrase "mathematical proof" conjures an image of the traditional two-column proof that is typical in high school geometry classrooms. This form of proof can be helpful for supporting students' efforts to develop a clear chain of statements, each relying on the prior, and for making sure that each statement is justified, as illustrated in Exhibit 3.11.

Exhibit 3.11. Grade 12 NAEP Geometry Proof Item



Given: C is the midpoint of $\overline{BE}$.
$\angle B$ and $\angle E$ are right angles.

Prove that $\overline{AC} \cong \overline{DC}$ and give a reason for each statement in your proof.

This item lends itself well to a two-column proof, particularly because it stipulates that a reason must be provided for each statement in the proof. One proof is as follows:

Statement	Reason
C is the midpoint of $\overline{BE}$	Given
$\angle B$ and $\angle E$ are right angles	Given
$\overline{BC} \cong \overline{EC}$	Definition of midpoint
$\angle B \cong \angle E$	Right angles are congruent
$\angle ACB \cong \angle DCE$	Vertical angles are congruent
$\triangle ACB \cong \triangle DCE$	Angle-Side-Angle (or Leg-Angle)
$\overline{AC} \cong \overline{DC}$	Corresponding parts of congruent triangles are congruent

Although this proof follows a typical form of school mathematics proof, there is nothing about the prompt that stipulates that the proof must occur in a two-column format. A narrative form of the proof in answer to the item in Exhibit 3.11 could also be appropriate, as seen below:

> The measures of $\angle BCA$ and $\angle ECD$ are equal because vertical angles have the same measure. We also know that the measures of $\angle B$ and $\angle E$ are the same because they are both right angles. Since C is the midpoint of $\overline{BE}$, $\overline{BC} \cong \overline{EC}$. So, by the angle-side-angle rule, triangle ACB is congruent to triangle DCE. Therefore, $\overline{AC} \cong \overline{DC}$ because corresponding parts of congruent triangles are congruent.

In addition to the various formats one can use to develop or present proofs, there are other ways of mathematically proving, disproving, or justifying a mathematical answer. These include developing deductive arguments, finding counterexamples, proving by exhaustion (i.e., verifying every possible case), and employing mathematical induction. Often, it may be easier to use a particular mode of argumentation based on the nature of the claim.

The process of refuting—demonstrating that a statement is false—is a key element of justification because conjecturing can produce both true and false statements. Students must understand that a single counterexample disproves a conjectured generalization.

An example of the value of finding a counterexample can be seen in the grade 12 algebra item in Exhibit 3.12. Here, one could identify a value for x that is, for instance, less than 5 but not also greater than –3 (e.g., $x = -10$). That single counterexample is sufficient to show that Dave's claim cannot be correct because $x = -10$ does not satisfy the statement $-3 < x < 5$.

Exhibit 3.12. Grade 12 NAEP Algebra Counterexample Item

Question A: If x is a real number, what are all values of x for which $x > -3$ <u>and</u> $x < 5$?

Question B: If x is a real number, what are all values of x for which $x > -3$ <u>or</u> $x < 5$?

Barbara said that the answers to the two questions above are different.

Dave said that the answers to the two questions above are the same.

Which student is correct?

○ Barbara ○ Dave

Explain why this student is correct. You may use words, symbols, or graphs in your explanation.

The questions at the start of the item in Exhibit 3.12 could be altered to give a grade 8 item:

Question A: If x is a number, what are all values of x for which $x \geq -3$?
Question B: If x is a number, what are all values of x for which $x > -3$?

The rest of the item would remain the same.

Similarly, only one counterexample is needed in the grade 8 Number Properties and Operations item shown in Illustration 3.13 (based on the item in Exhibit 3.13 in the Framework). Multiplying 6 by any real number less than 1 will yield a result less than 6, confirming Tracy's claim and refuting Pat's claim.

Illustration 3.13. Justifying and Proving Example: Confirming/Refuting Claims
based on **Exhibit 3.13**

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
12	Number Properties and Operations	Justifying and Proving	Num – 3.d	SCR

Tracy said, "I can multiply 6 by another number and get an answer that is smaller than 6."

Pat said, "No, you can't. Multiplying 6 by another number always makes the answer 6 or larger."

Who is correct? Give a reason for your answer.

Scoring Information

Key	Tracy is correct. Examples of correct reasons: • If you multiply by a number *smaller* than 1, the result is less than 6. • 6 × 0 = 0 • 6 × ½ = 3 • 6 × (-1) = -6

ALD Notes for Item Developers

Basic	The item could be revised to require the selection of the description for why the product of any whole number and 1/2 is less than the whole number, focusing on the meaning of multiplication, not the value of the product.
Proficient	The item assesses understanding and use of a counterexample to refute a claim.
Advanced	The item could be revised to provide a statement about multiplying 6 and any irrational number, which students need to justify or refute.

The item in this illustration is based on a NAEP item. The original version of this item appeared in the 1992 grade 12 NAEP Mathematics Assessment with NAEP Item ID 1992-12M14 #2 M054801.

Understanding that a single counterexample undermines a general claim is an important but difficult aspect of justification. Learning to search for counterexamples and explaining why they are justifications is only one aspect of refutation. Attempting to prove that a conjecture is false can also lead to the development of new insights or ideas, as well as to the formation of different conjectures that can then be explored, refuted, or proved.

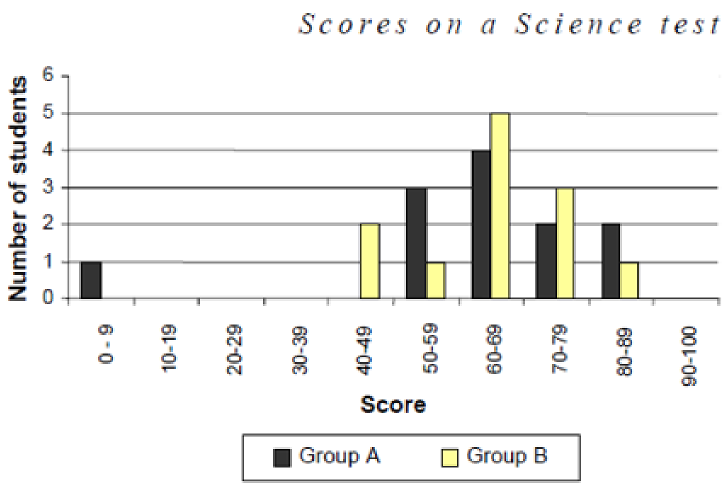
Explaining why something is true or not true is an important aspect of mathematical argumentation. However, not all mathematical arguments involve the NAEP Mathematical Practice of Justifying and Proving. Consider Illustration 3.14, a released 2006 PISA Data Analysis, Statistics, and Probability item (OECD, 2006). The item provides a data set and asks for a mathematical argument to counter a claim by a teacher that Group B did better. The requested mathematical argument is based on the single set of data represented in the graph, which is a specific instance and not a general case. Therefore, though the item in Illustration 3.14 may assess skill with inferential reasoning and identifying evidence for making a mathematical argument, it does not assess the NAEP Mathematical Practice of Justifying and Proving.

Illustration 3.14. Justifying and Proving Nonexample: Reasoning from Data

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
12	Data Analysis, Statistics, and Probability	Other	Data – 1.c	SCR

The diagram below shows the results on a Science test for two groups, labelled as Group A and Group B.

The mean score for Group A is 62.0 and the mean for Group B is 64.5. Students pass this test when their score is 50 or above.



Looking at the diagram, the teacher claims that Group B did better than Group A in this test.

The students in Group A don't agree with their teacher. They try to convince the teacher that Group B may not necessarily have done better.

Give one mathematical argument, using the graph, that the students in Group A could use.

Scoring Information

Key	One valid argument is given. Valid arguments could relate to the number of students passing, the disproportionate influence of the outlier, or the number of students with scores in the highest level. • More students in Group A than in Group B passed the test. • If you ignore the weakest Group A student, the students in Group A do better than those in Group B. • More Group A students than Group B students scored 80 or above.

The item in this illustration is based on a 2006 PISA item with Item ID M513Q01 – 0 1 9.

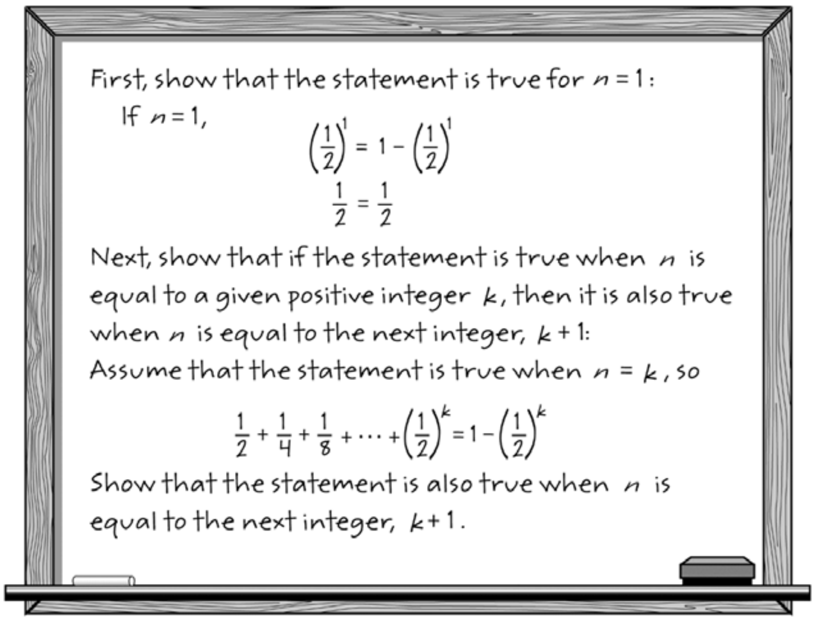
Some NAEP items require a specific mode of proof, such as the grade 12 Number Properties and Operations item in Exhibit 3.14.

Exhibit 3.14. Grade 12 NAEP Number Properties Mathematical Induction Item

A student was asked to use mathematical induction to prove the following statement.

$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\left(\frac{1}{2}\right)^n = 1-\left(\frac{1}{2}\right)^n$ for all positive integers n

The beginning of the student's proof is shown below.



Complete the student's proof by showing that if the statement is true when $n = k$, then it is also true when $n = k + 1$, where k is any positive integer.

Here, a student must use the tools of mathematical induction to complete the provided argument:

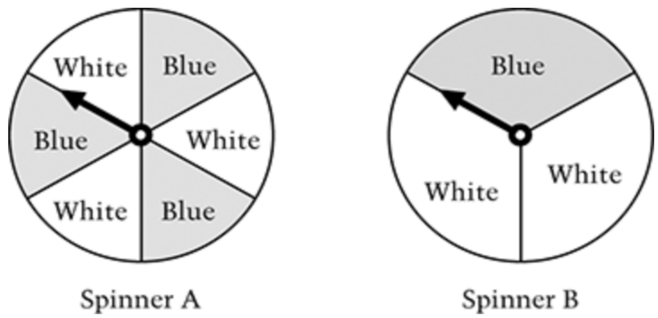
> For $n = k + 1$, $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\left(\frac{1}{2}\right)^{k+1}$ can be expressed as $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\left(\frac{1}{2}\right)^{k}+\left(\frac{1}{2}\right)^{k+1}$. We know from the above statement that $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\left(\frac{1}{2}\right)^{k}$ is equal to $1-\left(\frac{1}{2}\right)^{k}$, so substituting that yields $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\left(\frac{1}{2}\right)^{k+1} = 1-\left(\frac{1}{2}\right)^{k}+\left(\frac{1}{2}\right)^{k+1}$. Simplifying the expression on the right gives us $\frac{2^{k+1}-1}{2^{k+1}}$, or $1-\left(\frac{1}{2}\right)^{k+1}$.

Knowing a variety of approaches to generating a proof and knowing which one to select for a particular circumstance is an important aspect of justifying and proving.

Another element of justifying and proving is evaluating the validity of a purported proof. This involves not only deciding whether a proof is valid in terms of its conclusion, but also deciding whether a given proof relies on correct assumptions, makes use of merited conclusions and logic, and explains the entire statement or conclusion. These skills can be fostered by challenging students to judge the appropriateness of a given argument (e.g., a formal or informal proof;

Knuth, Choppin, & Bieda, 2009). Some NAEP items could be adjusted or expanded to include evaluating the justifications or proofs of others. For instance, the grade 8 NAEP item in Exhibit 3.15 addresses the question of maximizing the probability of landing on blue.

Exhibit 3.15. Grade 8 NAEP Probability Spinners Item



Lori has a choice of two spinners. She wants the one that gives her a greater probability of landing on blue.

Which spinner should she choose?

○ Spinner A ○ Spinner B

Explain why the spinner you chose gives Lori the greater probability of landing on blue.

The item in this illustration is based on a NAEP item. The original version of this item appeared in the 2011 grade 4 NAEP Mathematics Assessment with NAEP Item ID 2011-4M9 #15 M1609E1.

Asking students to explain *why* the spinner they chose gives Lori the greater probability of landing on blue foregrounds justifying; however, as given, a correct response to the item might be example-based rather than appealing to the general case. Students could also be given a version of this task in which other students' explanations for choosing Spinner A are provided, and then be asked which of the explanations is the most convincing to them and *why* it convinces them. Versions of the examples below might be offered as text, or by avatars, or through video.

1. Andreas says Spinner A has a greater chance for landing on blue because it has three blue sections and Spinner B only has one blue section.
2. Basil says that Spinner A will have a greater probability of landing on blue because the area of two of the blue sections on Spinner A is equal to the area of the one blue section on Spinner B.
3. Calista says that Spinner A has a greater chance of landing on blue because she tried it out. Calista spun each spinner 10 times. For Spinner A, the arrow fell on blue 6 times. For Spinner B, it only fell on blue 2 times.
4. Dora says that Spinner A will have a greater probability because it is one-half blue, but Spinner B is only one-third blue and one-half is more than one-third.

A task in which students evaluated these arguments would assess justifying and proving. It would provide students an opportunity to distinguish between an example-based justification (e.g., Calista's) and those based in what will generally be true of results from each spinner.

Engaging in justifying and proving is a way for students to explore why a particular assertion must be true. Granted, some proofs might only serve to verify the truth of a statement without helping students understand why; researchers refer to these as "proofs that prove" rather than "proofs that explain" (Hanna, 1990). Certainly not all proofs are explanatory, but in many cases, justifying or evaluating a given argument can help students understand why a conjecture is true. While investigating the reasons a conjecture might be true, students attend to particular features and consider relationships, examine multiple factors that are relevant to the problem statement, return to the meanings of terms and operations, or notice similarity or difference across cases. By exploring these factors, students gain new insight into the conjecture or deepen their understanding of fundamental mathematical ideas.

The grade 8 algebra item in Exhibit 3.16 foregrounds generalizing but could be revised into a justification task. In the item as given, the pattern that the number of diagonals d is equal to the number of sides $n - 3$ is readily apparent from the provided cases. However, adding a prompt asking *why* the equation $d = n - 3$ is a reasonable conjecture for any convex polygon would foreground justifying and proving. A valid justification might involve drawing a few cases, reasoning that from any given vertex one cannot draw a diagonal to itself and one cannot draw a diagonal to the two adjacent vertices (because this makes up two of the sides of the polygon), which means that three of the vertices cannot have diagonals drawn to them while the remaining vertices can.

Exhibit 3.16. Grade 8 NAEP Algebra Generalization Item

From any vertex of a 4-sided polygon, 1 diagonal can be drawn.

From any vertex of a 5-sided polygon, 2 diagonals can be drawn.

From any vertex of a 6-sided polygon, 3 diagonals can be drawn.

From any vertex of a 7-sided polygon, 4 diagonals can be drawn.

How many diagonals can be drawn from any vertex of a 20-sided polygon?

Answer:________________

The item in Exhibit 3.16 also could be revised into a task to justify why the total number of diagonals that can be drawn for any given convex polygon is $n(n - 3) / 2$. Justifying could take the form of first describing why the number of diagonals that can be drawn from a vertex is $n - 3$ (as above) and then reasoning that since there are n vertices, one could draw $n(n - 3)$ diagonals. However, this would mean that each diagonal would be drawn twice, to and from each vertex. Therefore, in order to avoid double-counting the diagonals, one must divide by 2, yielding the expression $n(n - 3)/2$. To further illustrate the difference between a proof that proves and one that explains, note that the expression for the total number of diagonals can also be proved by induction. Such a proof by induction would verify the statement without revealing why it is true.

Conversely, explaining one's reasoning for how a problem is solved by showing or describing steps in determining an answer is not sufficient for assessing the NAEP Mathematical Practice of Justifying and Proving. Consider the grade 8 item in Illustration 3.15. Responding to the item's prompts for a number of quarts and "how you found your answer" could be satisfied by a student writing down the calculations used to arrive at a conclusion (the number of quarts the student claims Tyler drinks in 7 days) using the given information (indisputable statements such as the information provided about Tyler's milk intake). However, the item does not explicitly call for the reasons why a student might multiply 24 by 7, then divide by 32. Furthermore, the item does not ask students to support why a proposed solution process would work for any problem of this type. For items that ask students to "show" or "explain" to measure justifying, a deductive argument about a general claim (in this case, a mathematical process) must be a part of the item's measurement intent.

Illustration 3.15. Justifying and Proving Nonexample: No General Claim

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
8	Measurement	Other	Meas – 2.b	SCR

Tyler drinks 24 fluid ounces of milk each day for 7 days. How many quarts of milk does he drink in the 7 days? Do not round your answer. (1 quart = 32 fluid ounces)

Answer: ____________________ quarts

Show how you found your answer.

Scoring Information	
Key	5.25 (or equivalent) quarts Possible work shown: $24 \times 7 = 168$ $168 \div 32 = 5.25$

The item in this illustration is based on a NAEP item. The original version of this item appeared in the 2013 grade 8 NAEP Mathematics Assessment with NAEP Item ID 2013-8M7 #9 M1687E1.

Justifying and proving can help students develop a new and deeper understanding of the mathematics content at hand. Making sense of others' justifications or proofs—and determining their validity—can help students generate new ideas, conjectures, and generalizations, or can support their efforts to develop a new theory to be tested. That is, justifying and proving is an important mode of communication. Proofs can reveal the tools, strategies, modes of thinking, and resources used by those who created them.

NAEP Mathematical Practice 4: Mathematical Modeling

Mathematical Modeling: Making sense of a scenario, identifying a problem to be solved, mathematizing it, applying the mathematization to reach a solution, and checking the viability of the solution in developmentally and mathematically appropriate ways.

Focus for Item Developers

The NAEP Mathematical Practice of Mathematical Modeling involves one or more components of the modeling cycle described by Garfunkel and Montgomery (2019, pp. 12–13). Components of the modeling cycle are referenced throughout this section, using descriptions and the letters from this list:

(a) identifying the problem;
(b) making assumptions that may simplify the problem and then identifying variables;
(c) mathematizing the situation;
(d) analyzing and assessing solutions;
(e) translating the solution(s) back into the real world and examining their feasibility, and, if not feasible, changing the simplifying assumptions and iterating the process;
(f) if there seems to be a feasible real-world solution, implementing the model; and
(g) reporting out results.

Each item associated with this practice should address more than one of the components.

Mathematical modeling involves student choice, including the assumptions made in the posing of answerable questions in an open-ended situation. The practice of modeling requires students to make sense of a scenario, identify a problem to be solved, mathematize it, and apply the mathematization to reach a solution and check the viability of the solution. Mathematical modeling also requires discussions and decisions about what is valuable (Burroughs & Carlson, 2019).

At an introductory level, modeling involves steps such as selecting and applying mathematical processes or expressing mathematical concepts and processes (such as mathematical operations) using visual, physical, or symbolic representations. At a more advanced level, a series of processes may be needed to mathematize a messy real-world situation prior to selecting and applying the mathematics. Follow-up work can involve analyzing and evaluating the results obtained from doing the mathematics. A full cycle in the mathematical modeling process includes: (a) identifying the problem; (b) making assumptions that often simplify the problem and then identifying variables; (c) mathematizing the situation; (d) analyzing and assessing solutions; and (e) translating the solution(s) back into the real world and examining their feasibility, and, if not feasible, changing the simplifying assumptions and iterating the process. Finally, if there seems to be a feasible real-world solution, there are two additional steps: (f) implementing the model; and (g) reporting out results (Garfunkel & Montgomery, 2019, pp. 12–13).

It is important to distinguish between the process of mathematical modeling and the noun "model," which is an object and a term sometimes used as a synonym for a mathematical representation. For example, when a line or other function is fitted to a bivariate scatterplot, the function is referred to as a model for the data, meaning a representation of the data. However, the practice of mathematical modeling involves far more than just using a representation. As previously described, mathematical modeling is a multistep process, which may involve aspects of representing, particularly building or interpreting a representation. However, the NAEP Mathematical Practice of Mathematical Modeling is distinct from that of Representing in that the use of representations in modeling is necessarily in service of the overarching purpose of identifying and finding solutions for problems in real-world situations. In assessment and item development, tasks that assess mathematical modeling may call upon the use of representations, but representing is not the primary focus of such a task. Rather, items assessing the NAEP Mathematical Practice of Mathematical Modeling focus on multiple steps of the cycle of mathematical modeling driven by that overarching purpose. For example, given an open-ended situation, students could generate questions they would need to explore or identify some assumptions as they begin the modeling process. In such scenarios, students would engage in the first two steps of the modeling process.

Although modeling tasks—especially separate aspects of the modeling process—could be posed to individual students, in the workplace mathematical modeling is often done in teams. The importance of preparing students to solve problems is regularly identified as a 21st-century skill. The U.S. Department of Labor, Office of Disability Employment Policy (2010), has noted:

> The ability to work as part of a team is one of the most important skills in today's job market. Employers are looking for workers who can contribute their own ideas, but also want people who can work with others to create and develop projects and plans. (p. 57)

In school mathematics, students already often work together in groups on mathematical tasks, and a mathematical modeling situation provides an inviting context for the use of collaborative tasks. The practice of mathematical modeling is also a natural place to use scenario-based tasks. Many of the sample tasks provided in this section could best be done by groups or pairs of students. When a task is worthy of group effort, the assessment could focus on group responses, solutions, and problem-solving activity. Such an assessment approach is central to the final practice of the NAEP Mathematics Framework, collaborative mathematics.

Scenario-based tasks are particularly useful in assessing student achievement in the practice of mathematical modeling. Consider the Lunch Problem scenario in Exhibit 3.17 (based on Garfunkel & Montgomery, 2019, pp. 38–42).

Exhibit 3.17. Grade 4 Example: Adaptation of GAIMME Lunch Problem Scenario

> *[Task is introduced through video: A school food service director states during the morning announcements that the school is planning a "Garden Bar" as an option for school lunch <video/image of a garden bar with a variety of fruits and vegetables> The director says, "The cafeteria staff and I would like your input, so we know that the fruits and vegetables included will be eaten. To assist us in our decision-making process, we are establishing a task force to help us gather your suggestions and will take your suggestions into account when making our decision."]*
>
> You volunteer for the task force.
>
> At the first meeting, the team works to determine what they need to know and how to go about gathering that information. Some of the questions your team identifies are:
>
> "How many students are in the school? Do students like some of these choices more than others? Do some of these choices cost more than others? If so, which ones might we have some left over, which might we run out of? Should the school's cost of these items be considered?"

From the scenario launch, several questions might be asked. Students who address these questions would be engaging in components (a) and (b) of the modeling cycle (identifying the problem and making assumptions).

Other tasks built from a similar scenario, about a pizza party for a grade 8 class, could be posed in different ways, depending on the aspect(s) of the modeling process being assessed. For example, grade 8 students could be given the open prompt: "How many and what types of pizzas should be ordered for an 8th grade party?" Some possible questions for students to address as they attempt to model this situation are: "How many students do we expect to feed? How can we find out what types of pizza they like? Should we survey some of the students? How do we decide who to survey? What sizes of pizzas should we order? What is the cost of each size of pizza?" Here students would need to devise a survey (identify the problem) and narrow down to choices of pizza and sizes of pizza (make assumptions; identify variables), and, as they begin to investigate costs of sizes and types of pizza, they would need to create estimates for the cost of the party (mathematize the situation; analyze and assess solutions).

At grade 12, a similar scenario-based open-ended task might include items based on a scenario such as: "What is the best type of computer for the school district to order for students to use in computer labs?" Some possible issues students may need to address as they attempt to model this situation are: "How many computers are needed in a school lab, and how do we know? Is there a break on cost if a large number of computers is purchased at the same time? Which types of classes will need access to the computers? What types of software will be needed for the classes? Do any of the companies offer deals for software along with the computer purchase? How much money can be spent per student?" There are many decisions to be made about what to include and what to assume to address this task. The problem also evokes initial mathematization processes when students ask questions like: “How much money per student?” or “Are there deals for software inclusion or a price break on a large order?”

Exhibit 3.18 is an example where some initial information is provided and students could work to develop a mathematical model (possibly in teams). The first three parts of the task are a scaffold to the modeling-heavy work of parts 4 and 5. Parts 3 and 4 engage students in aspects of modeling components (b), (c), and (d) when identifying variables, mathematizing situations, and analyzing and assessing solutions. Part 5 engages students in components (d), (e), and (g) of the modeling cycle.

Exhibit 3.18. Grade 12 Example: Modeling Income Tax Scenario

A state’s tax model is described below.

- Individuals with an income of \$10,000 or less per year pay no income tax.
- Individuals with income greater than \$10,000 per year pay a 6% tax on all income over \$10,000.

(1) What would a resident who made \$40,000 pay in tax? What percent of this resident’s total income is paid in tax?
(2) What would a resident who made \$50,000 pay in tax? What percent of this resident’s total income is paid in tax?
(3) Determine a method for calculating the percent of any resident’s total income that is paid in tax.
(4) Is there a highest percent of total income that a resident could pay in tax? Defend your position on this percent.
(5) The state is considering the new tax model described below:
- Individuals with an income of \$10,000 or less per year pay no income tax.
- Individuals with an income greater than \$10,000 per year
  - pay 5% on all income over \$10,000 up to \$50,000, and
  - pay 7% on all income over \$50,000.

Explain whether or not the new tax model benefits individuals in the state who pay income tax. As part of your response, compare the new tax model to the existing tax model.

Access to digital tools, such as equation editors, graphing tools, and spreadsheet tools, would be important in the assessment of students’ modeling practices on tasks like Exhibit 3.18. For example, in parts 3 and 4, the percent income paid in tax can be expressed as the ratio of tax T to income I, or T/I (identify variables). When students compute the tax on income I, with the given 6% rate after the first \$10,000 of income, they arrive at $T = 0.06(I - \$10{,}000)$ (mathematize the situation). A symbolic model for the percent income paid in tax could be $T/I = 0.06(I - 10{,}000)/I$. To answer questions about the highest possible tax rate, students could create a graphical model of the percent income paid in tax as a function of income, I. The mathematization process for this task starts with decisions about using ratios and percent and then could evolve to developing an algebraic expression to model the percent income paid in tax or even a graph of the percent income paid in tax as a function of income (analyzing and assessing the solution). Modeling carries through to parts 4 and 5 as students compare the new model to the original model. This comparison could be explored through the use of a spreadsheet tool that allows students to rapidly compute the total tax for a given income based on each model. In part 5, the item includes component (g) of the modeling cycle with students reporting out on whether or not the new tax model is recommended.

Modeling processes also often arise in data analysis and statistics. The task in Exhibit 3.19 is an example taken from the online bank of tasks available from Levels of Conceptual Understanding in Statistics (LOCUS, 2019).

Exhibit 3.19. Grade 8 LOCUS Data Modeling Task

The student council members at a large middle school have been asked to recommend an activity to be added to physical education classes next year. They decide to survey 100 students and ask them to choose their favorite among the following activities: kickball, tennis, yoga, or dance. (a) What question should be asked on the survey? Write the question as it would appear on the survey. (b) Describe the process you would use to select a sample of 100 students to answer your question. (c) Create a table or graph summarizing possible responses from the survey. The table or graph should be reasonable for this situation. (d) What activity should the student council recommend be added to physical education classes next year? Justify your choice based on your answer to part (c).

As posed, this task covers the complete modeling cycle from (a) to (g) and closely follows the statistical investigation process as outlined by Bargagliotti and colleagues (2020): identifying a statistical question for investigation, gathering appropriate data, analyzing the data, and communicating the results. The task assesses several content objectives in the data analysis, statistics, and probability area, including posing a statistical question, addressing issues of bias in surveys, and creating tables and graphical representations of data. Though the task as written addresses a full modeling cycle, some parts could be supplied to students and then students could be asked to engage in a narrower aspect of the modeling process.

The mathematical literacy–focused modeling task in Illustration 3.16 was adapted from a water crisis task developed for use with teachers (Aguirre, Anhalt, Cortez, Turner, & Simic-Muller, 2019). In this task, students are asked to think as a member of a team working to solve a problem. They are not asked to work through the entire modeling process, which would take more time than a scenario-based task would allow. Instead, the content is scaffolded to provide access to aspects of the modeling process as a path to a possible solution to the question posed. Through the task, students need to determine variables of interest, analyze the model presented for the community, and translate this model to the science club members' town.

Illustration 3.16. Mathematical Modeling Example: Scenario-Based Task with a Mathematical Literacy Context

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
12	Data Analysis, Statistics, and Probability	Mathematical Modeling	Data – 3.c Data – 3.d	ECR

Members of the science club saw a news clip about a water crisis.

Science club members began the process of answering the question: "How many bottles of water would be needed to supply drinking water to each person in our town should our water supply become harmful to drink?" They create a mathematical model to help them answer this question.

1) Write 5 things that the science club members need to consider as they work to determine the number of bottles of water their community would need in a water crisis.

2) The science club members want to learn what they can from the water crisis in the news. They read that from May 2018 through the end of August 2019, one company donated over 6.5 million bottles of water to the community.
   - Write **two** questions to which the members need answers so they can determine how many bottles of water the community in the news actually needed.
   - Explain how knowing answers to these two questions will help the science club members in the process of answering their question.

3) There were about 96,000 residents in the community in the news in 2018. To meet the drinking-water needs of that community required more than 25 million 0.5-liter bottles. The science club members' town has a population of about 4,000 people.

   Estimate the number of 0.5-liter bottles needed to meet the town's drinking water needs. Justify your response.

A real-world situation such as a water crisis provides a wealth of material from which a modeling task can be built. It calls on students to determine and apply relevant information to solve a problem.

Not all representations of mathematical thought address a component of the modeling process. The grade 8 item in Illustration 3.17 (from which Exhibit 3.9 was adapted) asks students to list all of the possible outcomes of flipping three coins. Absent from this released item are key aspects of mathematical modeling discussed previously, including student choice and discussions and decisions about what is valuable. To address the practice of mathematical modeling in a coin-flipping situation, a more open-ended task could be developed in which student thinking is in service of the overarching purpose of identifying and determining a solution for a problem in a real-world situation. For example, an item could state, "Someone puts an unfair coin in a stack with 5 fair coins. All of the coins look identical. Create a process that uses only coin flips and mathematics for determining which of the coins is the unfair coin."

Illustration 3.17. Modeling Nonexample: List Possibilities without Connection to the Modeling Process

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
8	Data Analysis, Statistics, and Probability	Other	Data – 4.e	SCR

A nickel, a dime, and a quarter are flipped at the same time. Each coin can land either heads up (H) or tails up (T). List all the different possible outcomes for this event in the chart below. The list has been started for you.

Nickel	Dime	Quarter
H	H	H
H	H	T

Scoring Information

Key	HHH – given HHT – given HTT HTH THT TTH TTT THH

The item in this illustration is based on a NAEP item. The original version of this item appeared in the 2013 grade 8 NAEP Mathematics Assessment with NAEP Item ID 2013-8M3 #2 M1499E1.

As mentioned, the terms "represent" and "model" are often used interchangeably. For the purposes of NAEP, creating or using a mathematical representation may indicate the practice of representing, but may or may not be invoked by the practice of mathematical modeling. For example, "Use an algebraic model to estimate height" was the description in the NAEP Questions Tool (NCES, n.d.) of the item shown in Illustration 3.18. In this item, students use a given representation to solve a problem, assessing the NAEP Mathematical Practice of Representing. However, the NAEP Practice of Mathematical Modeling is not assessed. In part, this is because students are not asked to situate the equation within the modeling cycle.

Illustration 3.18. Modeling Nonexample/Representing Example: Evaluating a Formula to Answer a Question

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
8	Algebra	Representing	Alg – 4.d	SR – MC

Archaeologists measure the lengths of certain bones to estimate a dinosaur's height. When the length t of the tibia, or leg bone, is known, a dinosaur's height h can be estimated by the following formula, where t and h are in centimeters.

$$h = 73 + 2.5t$$

If the length of the tibia of a certain dinosaur is 400 centimeters, what is its estimated height in centimeters?

A. 402.5

B. 473

C. 475.5

D. 1,000

E. 1,073

Scoring Information	
Key	E. 1,073

The item in this illustration is based on a NAEP item. The original version of this item appeared in the 2013 grade 8 NAEP Mathematics Assessment with NAEP Item ID 2013-8M3 #14 M151101.

Note that items developed using a definition of modeling other than that described in the Framework may not assess the NAEP Mathematical Practice of Mathematical Modeling. Attending to the requirements for representing and modeling will be useful in item development and will allow for distinguishing NAEP Mathematical Practices assessment intent.

Illustration 3.19 shows the first part of a released PARCC item. When considering only this part of the item, the NAEP Mathematical Practice of Mathematical Modeling is not assessed.

The request for a "model" in the item in Illustration 3.19 calls for students to create a symbolic representation of the relationships among costs without requiring substantive engagement in the modeling cycle. Components (a), (b), and (c) of the modeling cycle are provided in the item stem; components (d) and (e) are neither required by nor used as scaffolding within the item; and no opportunity for components (f) or (g) exists in the item as written. Attention to the nuance in the assessment of modeling is included in this document because item writers may work on item development for several different assessments at the same time. This illustration is intended to highlight the distinctions between assessment intent for NAEP Mathematical Modeling items and that for other assessments.

Illustration 3.19. Modeling Nonexample/Representing and Mathematical Literacy Example

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
12	Algebra	Representing	Alg – 4.d	ECR

A family compares the costs of renting a truck from two different companies for its 2-day move to another state. The costs are shown in the table.

Truck Rental Costs

Item	Company X	Company Y
base rental charge	\$29.95 per day	\$19.95 per day
mileage charge	59 cents per mile	79 cents per mile
drop-off charge	\$150	included
insurance	\$18 per day	\$26 per day

Part A

Create a model that can be used to determine the rental cost of each truck for the 2-day move. Describe the process you used to determine your model.

Use your model to determine the number of miles when the rental costs of the two trucks will be equal.

Enter your answers in the space provided.

- Math symbols
- Relations
- Geometry
- Groups
- Trigonometry
- Statistics
- Greek

Scoring Information

Key	Student response includes the following 4 elements. • Valid definition of variables • Valid model of the rental costs for Company X • Valid model of the rental costs for Company Y • Correct number of miles when the rental costs of the two trucks will be equal

The item in this illustration is based on a 2018 PARCC item with Item ID VH145748, aligned to evidence statement HS-D.CCR.

NAEP Mathematical Practice 5: Collaborative Mathematics

> *Collaborative Mathematics: The social enterprise of doing mathematics with others through discussion and collaborative problem solving whereby ideas are offered, debated, connected, and built-upon toward solution and shared understanding. Collaborative mathematics involves joint thinking among individuals toward the construction of a problem solution in developmentally and mathematically appropriate ways.*

Focus for Item Developers

The NAEP Mathematical Practice of Collaborative Mathematics involves the engagement of mathematical knowledge and skills within a collaborative context. The three measurable skills associated with this practice are:

- attending to and making sense of the mathematical contributions of others,
- evaluating the mathematical merit of the contributions of others, and
- responding productively to others' mathematical ideas.

Each item associated with this practice should address one or more of these skills.

Collaborative mathematics in the world of work refers to the talk and actions people engage in with one another as they participate in a necessary collaboration, where the mathematical task is too complex or messy for an individual to meet its demands alone (Fiore et al., 2017). The degree of complexity is different for collaborative mathematics tasks used in schools and assessments. It is true that tasks are designed to require collaboration (e.g., multiple parts, multiple roles, multiple strategies, comparisons of strategies), but it is not necessarily true that the mathematics is too complex or messy for one student; rather, the task may be such that it is designed to require multiple people.

As a practice, collaborative mathematics exists alongside other mathematical practices. That is, as students work together toward a shared goal, they may also engage in representing, abstracting and generalizing, justifying and proving, and mathematical modeling. Assessing collaborative mathematics requires developing items that foreground and require the doing of mathematics collaboratively, engaging processes that are fundamentally about *joint thinking* (Teasley & Roschelle, 1993). Collectively, these processes include sharing ideas with others; attending to and making sense of the mathematical contributions of others; evaluating the merit of others' ideas through agreement or disagreement; and productively responding to others' ideas through building on or extending ideas and connecting or generalizing across ideas.

Collaborative mathematics processes are largely understood as discursive in nature and occurring through social interaction during mathematical activity. NCTM's policy documents reflect a long-standing focus on discourse and communication. Beginning with the Mathematics as Communication standard (NCTM, 1989) and attention to discourse (NCTM, 1991), mathematics educators have argued that when students write and talk about their thinking, not only do they clarify their own ideas, but they also offer valuable information for assessment.

Given the discursive nature of collaborative mathematics, NAEP Mathematics Assessment items that measure collaborative processes should likewise be discursive in nature, offering students examples of social interaction or imagined utterances around mathematics to which they are tasked to respond in key ways. These include being asked to make sense of others' thinking, express and defend agreement or disagreement, and extend an idea. Tasks might also be genuinely collaborative in nature, asking assessed students to work together in a team during the assessment, such as on a mathematical modeling task.

The discursive nature of collaborative mathematics also means that it is a highly contextualized activity, tied to cultural ways of working together both in and out of the classroom. As stated in the opening of this chapter, while state standards have long included mathematical practices, and collaboration among students has long been emphasized, instruction that engages students in mathematical practices generally, and through collaborative activity in particular, may not yet be pervasive. Without careful attention to opportunities to learn, the assessment may privilege particular out-of-school cultural repertoires for collaboration, particularly around critique.

The assessment of collaborative activity is not new. The Programme for International Student Assessment (PISA), for example, assesses collaborative problem solving, defined as:

> the capacity of an individual to effectively engage in a process whereby two or more agents attempt to solve a problem by sharing the understanding and effort required to come to a solution and pooling their knowledge, skills, and efforts to reach that solution. (OECD, 2017, p. 6)

As illustrated in the components from a PISA scenario-based collaborative problem-solving task (Exhibits 3.20 and 3.21), the task structure involves a dialogue between a team of avatars and the assessed student. The problem task is on the right of the screen, while the running dialogue is on the left (Exhibit 3.20). The assessed student is to choose a discursive response to productively move the collaboration forward. In the example offered in the subsequent screenshots in Exhibit 3.21, one can see that the components of the task emerge as interactional contributions are offered by each avatar (e.g., "Brad") and the assessed student ("you") through item response choices.

Exhibit 3.20. Example PISA Collaborative Problem-Solving Item

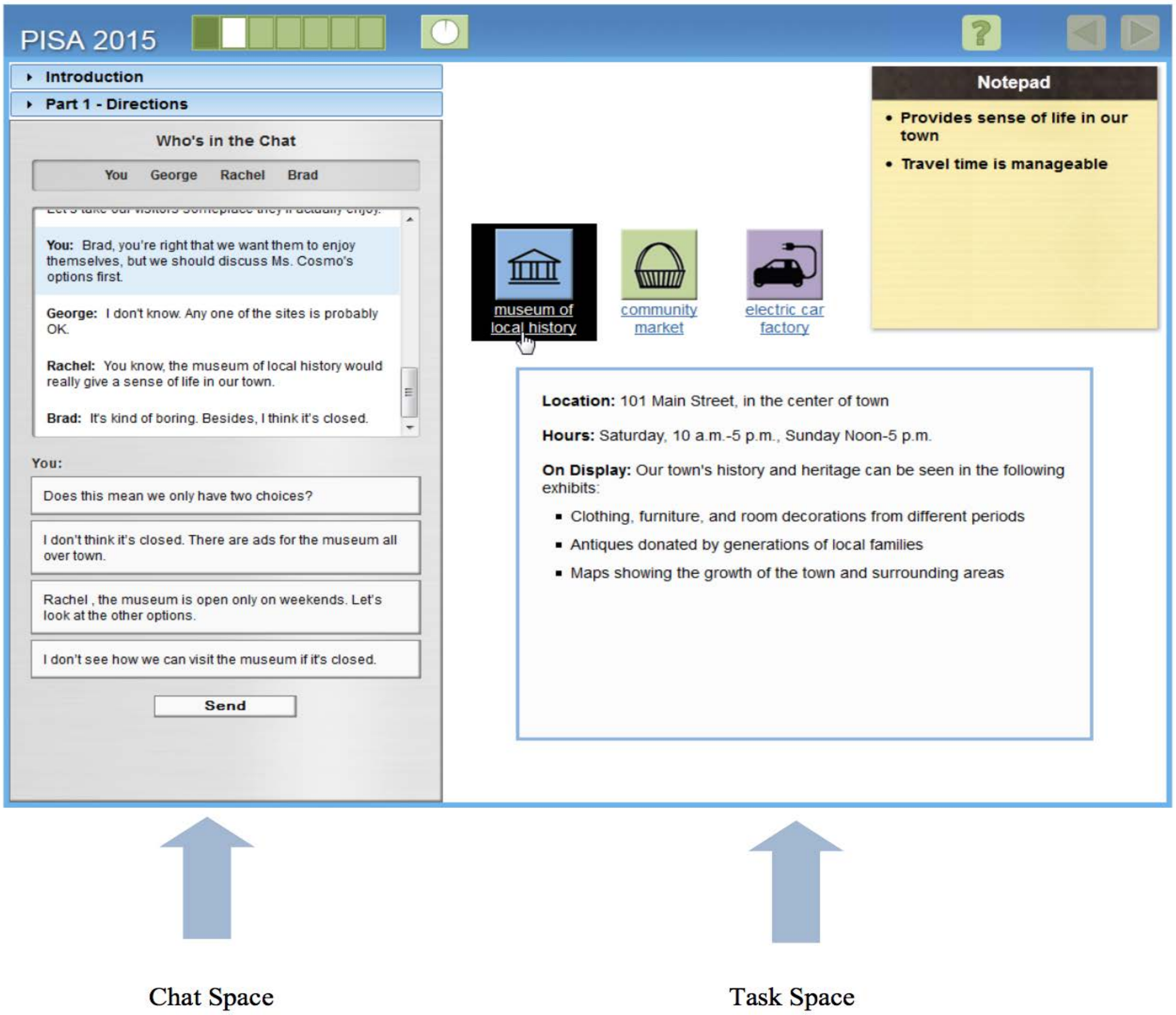
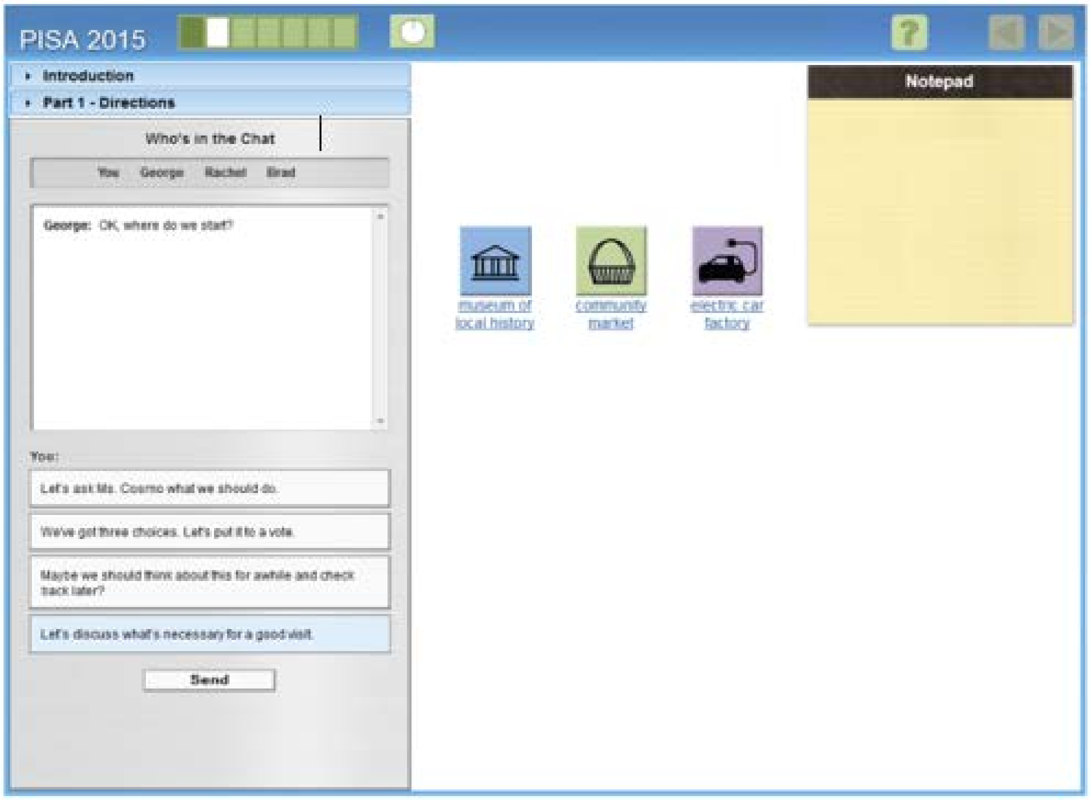
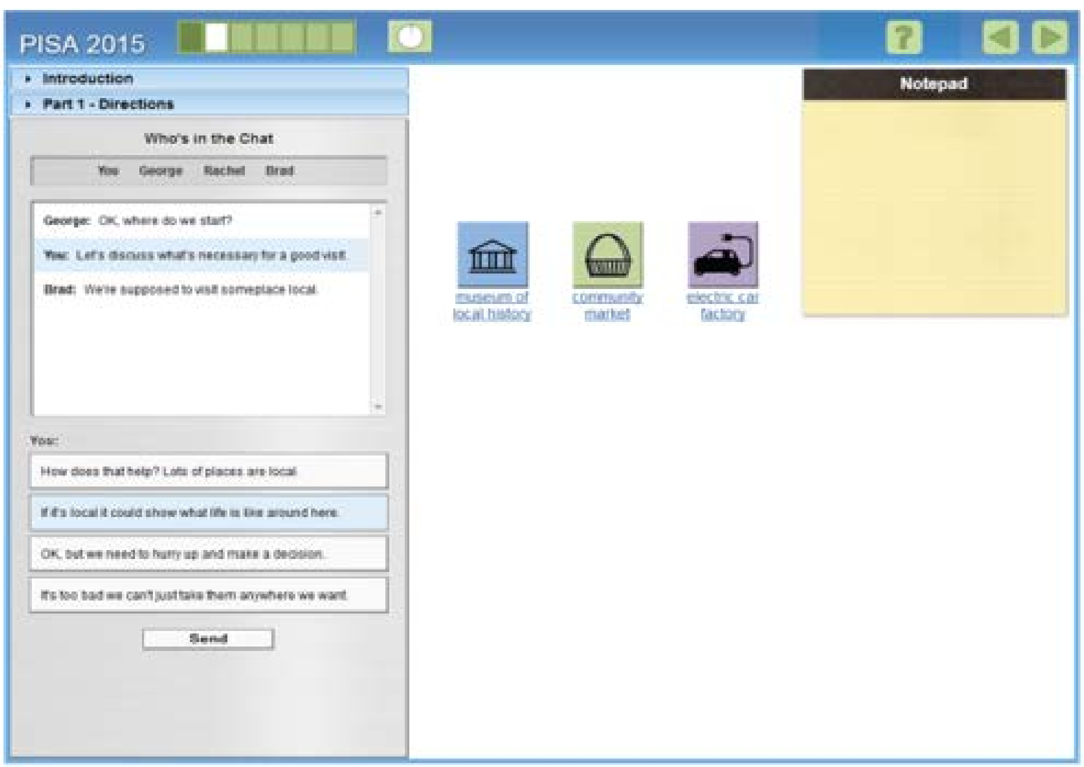


Exhibit 3.21. Example PISA Collaborative Problem-Solving Interaction



Brad mentions that the group is supposed to visit someplace local.



While PISA collaborative problem-solving items are helpful in highlighting discursive assessment, PISA items are not specifically focused on mathematics. Rather, PISA assesses three generic collaborative problem-solving competencies: establishing and maintaining a shared understanding; taking appropriate action to solve the problem; and establishing and maintaining team organization. Additionally, PISA's collaborative problem-solving items are intended to assess problem-solving competencies such as exploring and understanding; representing and formulating; planning and executing; and monitoring and reflecting. These competencies are assessed at varying levels of the collaborative skill (OECD, 2017).

Some of these competencies may apply to collaborative mathematics, but the aim for NAEP is to assess the collaborative processes involved in *mathematics* in particular. The following sections describe three measurable skills involved in collaborative mathematics:

- attending to and making sense of the mathematical contributions of others,
- evaluating the mathematical merit of the contributions of others, and
- responding productively to others' mathematical ideas.

Collaborative mathematics items may assess one or more of these aspects as a student engages with others during the assessment (e.g., a human, a computer-based avatar, or named characters introduced in the item stem). Measurement targets are at the intersection of the assessed student's cognitive and social processes within a collaborative context. For the NAEP Mathematics Assessment, the collaborative process will most often begin for the assessed student after an initial presentation of mathematical context and content.

Features of items can include negotiating mathematical ideas through such activities as:

- expressing agreement, disagreement, or uncertainty;
- requesting clarification;
- elaborating on or revoicing others' ideas;
- identifying conflicts or gaps in mathematical thinking; and
- revising one's own thinking.

Negotiation may or may not entail conflict, but it does entail the processes through which team members accommodate and resolve differences on the way to coming to agreement (Dillenbourg & Baker, 1996; Fiore et al., 2017; Hesse, Care, Buder, Sassenberg, & Griffin, 2015). The agents involved in the collaboration coordinate their item- or task-relevant interactions, developing shared understandings, and constructing solutions.

Items can also involve establishing and maintaining team discourse relevant to the item or task at hand by (Flor, Yoon, Hao, Liu, & von Davier, 2016; Hao, Liu, von Davier, Kyllonen, & Kitchen, 2016):

- identifying goals;
- communicating next steps;
- evaluating teamwork; and
- checking understanding.

The structure of collaborative items or tasks allows for interaction between team members in a way that informs their thinking in item- or task-relevant ways (Dillenbourg, 1999). Individual thinking "can be inferred from the actions performed by the individual, communications made to others, intermediate and final products of the problem-solving tasks, and open-ended reflections on problem-solving representations and activities" (OECD, 2017, p. 135).

The assessed student and agents with whom the student engages can take on different roles in the collaborative process. A student may take on multiple roles within one interchange, depending on how an action is structured (Chiu, 2000):

- *Facilitator*: guides the group, helping to maintain focus and productivity
- *Proposer*: communicates claims
- *Supporter*: communicates agreement
- *Critic:* communicates disagreement
- *Recorder*: synthesizes group communications

For each of the three measurable collaborative skills described in this section, the table in Illustration 3.20 lists potential student actions and associated student roles.

Illustration 3.20. Potential Student Actions and Roles Associated with the Three Measurable Collaborative Mathematics Skills

Collaborative Mathematics Skill	Potential Student Actions	Potential Student Roles
Attending to and making sense of the mathematical contributions of others	Student asks the teammate to repeat a statement.	Facilitator
	Student asks the teammate to clarify a statement.	Facilitator
	Student rephrases/completes the teammate's statement.	Recorder
	Student identifies the goal of the conversation.	Facilitator/Recorder
	Student expresses confusion/frustration or lack of understanding.	Facilitator/Critic
	Student expresses progress in understanding.	Facilitator/Supporter
	Student checks on understanding.	Facilitator
Evaluating the mathematical merit of the contributions of others	Student expresses agreement with teammates.	Supporter
	Student expresses disagreement with teammates.	Critic
	Student expresses uncertainty of agreement or disagreement.	Supporter/Critic
	Student identifies a conflict in their own idea and the teammate's idea.	Critic
	Student uses relevant evidence to point out some gap in the teammate's statement.	Critic
	Student expresses what is missing in the teamwork to solve the problem.	Critic
	Student evaluates whether certain group contribution is useful or not for the problem solving.	Supporter/Critic
	Student points out some gap in a group decision.	Facilitator/Critic
	Student identifies a problem in problem solving.	Facilitator/Critic
Responding productively to others' mathematical ideas	Student elaborates on their own statement.	Supporter
	Student changes their own idea after listening to the teammate's reasoning.	Supporter
	Student suggests the next step for the group to take.	Proposer
	Student reflects on what the group did.	Recorder

Other aspects of the assessment of collaborative mathematics should be informed by results from the special studies described in Appendix E. These aspects include, but are not limited to, whether the agents with whom the assessed student engages are human or computer-based, process data collected, and methods of data collection.

Attending to and Making Sense of the Mathematical Contributions of Others

Collaborative mathematics begins with the sharing of ideas in the form of a conjecture or other contribution that is meant to be communicated to others. A first joint act is made up of both this sharing and how others attend to the conjecture and make sense of it (Forman, Larreamendy-Joerns, Stein, & Brown, 1998). To do so, students must establish a shared understanding about what the problem is and how the problem is being interpreted (Lerman, 1996).

While classroom studies document the importance of making sense of peers' ideas during collaborative mathematics activity, most research on the discursive processes in making sense of student thinking has looked at teacher talk moves rather than student talk moves (Chapin, O'Connor, O'Connor, & Anderson, 2009). These moves are nevertheless relevant in framing how students make sense of one another's mathematical thinking. For example, people *elicit* and *probe* ideas. Individuals then express and check personal understanding of another's thinking by repeating or *revoicing* the idea (Enyedy et al., 2008). During a collaborative mathematics assessment task, students can elicit, probe, and revoice peers' ideas to demonstrate and check for understanding.

Negotiation skills such as requesting clarification and revoicing others' ideas have been shown to be a sign of effective collaboration (Hao et al., 2016). Revoicing is a particularly powerful discursive opportunity to assess whether a student has understood the mathematical contribution of others. Revoicing is defined as "when one person re-utters another's contribution through the use of repetition, expansion, or rephrasing" (Enyedy et al., 2008, p. 135). From an assessment perspective, students can be asked to revoice (or put into their own words) the expressed mathematical ideas of another student/an avatar, or to justify its mathematical appropriateness.

The item in Illustration 3.21 is adapted from Exhibit 3.12 in the Framework (discussed previously in the justifying and proving section of this chapter). In both the original item and the adapted item, students are asked to make sense of the mathematical contributions of others as they evaluate the correctness of given statements. The adapted item includes an opportunity for students to consider original and revoiced statements. Also note that the names in the adapted item are different from those used in the original item. This change is to increase the diversity of the names contained in exhibits and illustrations throughout this document.

Illustration 3.21. Collaborative Mathematics Example: Revoicing
adapted from **Exhibit 3.12**

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
12	Algebra	Collaborative Mathematics	Alg – 4.c	SCR – composite
Kala and Samir discussed the two questions shown. Question A: If x is a real number, what are all the values of x for which $x > -3$ <u>and</u> $x < 5$? Question B: If x is a real number, what are all the values of x for which $x > -3$ <u>or</u> $x < 5$? Kala stated that the answers to the two questions are different. Samir agreed with Kala, stating that the compound inequality $-3 < x < 5$ and the compound inequality $x > -3$ or $x < 5$ have different solution sets. Explain how Kala's statement and Samir's statement are equivalent. You may use words, symbols, or graphs in your explanation.				

The item in this illustration is adapted from a NAEP item. The original version of this item appeared in the 2013 grade 12 NAEP Mathematics Assessment with NAEP Item ID 2013-12M99 #1 M1934E1.

Evaluating the Mathematical Merit of the Contributions of Others

Once students attend to and make sense of the thinking of others, they must evaluate the mathematical reasonableness of their peers' mathematical contributions. Generally, students express their evaluation of the mathematical reasonableness of an idea through agreement or disagreement, including some explanation or justification. Agreeing or disagreeing emerges out of shared understanding (Nathan, Eilam, & Kim, 2007). This skill is critical to the development of productive mathematical argumentation. Experimental and classroom studies have found that students' ideas can be evaluated and become influential due to issues of status or authority rather than mathematics sense-making (Cohen & Lotan, 1997; Engle, Langer-Osuna, & McKinney de Royston, 2014).

Expressing agreement or disagreement is a negotiating skill associated with collaboration (Hao et al., 2016). Exhibit 3.22 shows a grade 4 SBAC (2018) item suited to assess the collaborative skill of evaluating the mathematical merit of the contributions of others. In the item, the assessed student is offered a strategy for solving a problem by an imagined student, Connor. The assessed student is asked to evaluate Connor's stated strategy and decide whether or not he is correct and why. When answering this item correctly, the assessed student takes on the role of supporter, as the correct response indicates agreement with Connor's statement along with an explanation for the agreement (Chiu, 2000). Digitally based administration of this and similar items could provide the assessed student the opportunity to read or hear (through voiceover) Connor's own utterances, make sense of Connor's thinking, and then choose an evaluation with explanation. Note that hearing Connor's words does not make the item collaborative. Collaborative mathematics is tied to the nature of the item, which begins with a collaborative situation and illustrates a very basic instance of looking into another person's strategy, requiring students to attend to and make sense of Connor's mathematical contribution and evaluate the mathematical merit of Connor's claim.

Exhibit 3.22. Adapted Grade 4 SBAC Number Properties Collaborative Mathematics Item

> Together, you and Connor are finding 8×16.
> Connor says, "We can find the product if we multiply 8 and 15 and then add 8."
> Which sentence could you say to Connor to **best** explain that his statement is correct or incorrect?
>
> A. I think you are incorrect, because we should add 16 instead of 8.
> B. I think you are correct, because 15 is an easier number to multiply by than 16.
> C. I think you are correct, because 8×16 is the same as 15 groups of 8, plus 1 group of 8.
> D. I think you are incorrect, because 8×16 is the same as 4 groups of 8, plus 4 groups of 8.

Another negotiating skill associated with collaboration is the use of relevant evidence to point out a gap in a teammate's statement (Hao et al., 2016). Illustration 3.22, based on Exhibit 3.23, shows another grade 4 item from the SBAC collection. Like the previous item, the item begins with a collaborative situation within which the assessed student is offered a glimpse into the thinking of an imagined peer, Jose. Here, Jose offers a conjecture about number. The assessed student is asked to critique Jose's conjecture by offering a counterexample that proves Jose's statement false. In this item, the assessed student takes on the role of critic, as the item stem indicates disagreement with Jose's statement and the completion components of the item support the disagreement (Chiu, 2000). A digitally based assessment means the assessed student could have the opportunity to read or hear (through voiceover) Jose's own utterance, make sense of Jose's thinking, and then complete a sentence that shows why Jose's statement is false. Although the item tells the student that Jose's statement is incorrect, the assessed student needs to understand Jose's statement before responding. The item also addresses the practice of justifying and proving, through the required completion of a counterexample to refute Jose's statement.

Illustration 3.22. Multi-Practice Example: Collaborative Mathematics with Justifying and Proving *based on* **Exhibit 3.23**

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
4	Number Properties and Operations	Justifying and Proving Collaborative Mathematics	Num – 5.b	SR – matching

You and Jose talk about the number of factors all whole numbers have.

Jose says that all whole numbers except 1 have an even number of factors because factors always come in pairs.

Jose's statement is incorrect.

Complete the sentences to help Jose see that his statement is not always correct. Drag numbers into the empty boxes to complete the sentences.

1
2
3
4
5
6
7
8
9
10

What about the number ☐?

It has ☐ factors.

Scoring Information

Key	4; 3 OR 9; 3

ALD Notes for Item Developers

Notes for Basic and Advanced address both Collaborative Mathematics and Justifying and Proving.

Basic	The item could be revised to decrease collaborative demands. For example, the revision could provide a true statement about factors and ask for a revoiced version of the statement. The item could be revised to decrease justifying and proving demands. For example, the revision could require the selection of a provided description for why 4 is a factor of itself, focusing on the meaning of factor as applicable to any whole number.
Proficient	The item assesses consideration of a mathematical statement made by another in concert with understanding of factors to complete an argument that refutes the given statement.
Advanced	The item could be revised to increase collaborative demands. For example, the revision could provide multiple statements from additional peers about factors across which making connections is required. The item could be revised to increase justifying and proving demands. For example, the revision could require the determination of the validity of Jose's statement along with completion of sentences that use the definition of factor to justify or refute Jose's statement.

The item in this illustration is based on an SBAC item with Item ID 3322, aligned to CCSS-M objective 4.OA.B.4.

The item in Illustration 3.22 involves the NAEP Mathematical Practices of Collaborative Mathematics and Justifying and Proving. As stated previously, the collaborative practice is often intertwined with other mathematical practices in the development of an item, but it should be possible to identify a primary practice focus. It is at the developer's discretion to determine which practice should be indicated as the primary practice. For this item, the practice focus is justifying and proving because the counterexample is fundamental to the completion of the item. The situating of the item in a collaborative context is not a requirement for arriving at the counterexample.

Consider, again, Illustration 3.13, a grade 12 NAEP Mathematics Assessment item also suited to assess collaborative mathematics. In the item, the assessed student is given an exchange by two imagined students, Tracy and Pat. That is, the assessment happens in the context of examining the justifying activity of Pat. Tracy offers a conjecture about which Pat expresses and explains disagreement. Assessed students are asked to evaluate these utterances and decide which is correct and to explain their evaluation. Again, an assessed student has the opportunity to read or hear (through voiceover) Tracy and Pat's own utterances. This conversational format is preferable to an offer of paraphrased positions that the assessed student is tasked to evaluate.

Responding Productively to Others' Mathematical Ideas

A third mathematics-specific collective process involves responding productively to others' mathematical ideas. In particular, students learn to build on, extend, and connect across mathematical ideas. These discursive acts depend and build on the acts of making sense of and evaluating others' mathematical thinking. Once a shared mathematical idea is understood, students can further contribute to the mathematical discussion by acting upon those shared ideas. Connecting across students' mathematical ideas is a core discursive component of productive collaborative mathematics (Stein, Engle, Smith, & Hughes, 2008). By connecting ideas, students are able to notice and explain how two seemingly different strategies hold the same mathematical ideas. Students also build on or extend an idea through new examples, next steps, or logical deductions.

The grade 4 item in Illustration 3.23a has potential, but, as written, does not assess the practice of collaborative mathematics. The established context is not inherently collaborative, except for the fact that students are asked to make sense of Mark's nonstandard first step and provide guidance for the next step.

While the item in Illustration 3.23a does not assess collaborative mathematics, it does assess the NAEP Mathematical Practice of Representing. In the item, students are presented with a symbolic representation of subtraction with which they engage as they consider the verbal representation of Mark's first step and ways of representing a next step in the solution process. That is, as they complete the item, students use and interpret presented representations.

Illustration 3.23a. Potential to Be a Collaborative Mathematics Example

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
4	Number Properties and Operations	Representing	Num – 3.a	SCR – composite

$$\begin{array}{r} 143 \\ -\ 48 \\ \hline \end{array}$$

Mark needs to solve the problem shown.

He will solve the problem in two steps.

First, Mark subtracts 43 from 143.

What does Mark need to do next to complete the problem?

What is the answer to **143 – 48** ?

Scoring Information

Key	a) Subtract 5 from 100. OR Subtract 5 from the result found in step 1. b) 95
Correct	Both parts correct
Partial	Part a) correct only OR Part b) correct only
Incorrect	Incorrect response

The item in this illustration is based on a NAEP item. The original version of this item appeared in the 2017 grade 4 NAEP Mathematics Assessment with NAEP Item ID 2017-4M1 #8 M3744E0.

The item in Illustration 3.23b provides a revision that places the mathematical action in an inherently collaborative context. In this item, the assessed student takes on the role of proposer, offering a next step in the subtraction process (Chiu, 2000). The collaborative demands of the revision could be increased by programming a computer-based agent to offer steps in the subtraction process until the team of Mark and the assessed student arrives at the difference.

Illustration 3.23b. Collaborative Mathematics Example: Determining a Difference Collaboratively

<table>
<tr><td colspan="2">You and Mark take turns suggesting steps to find the difference shown.

$$\begin{array}{r} 143 \\ -\ 48 \\ \hline \end{array}$$
Mark’s first step is to subtract 43 from 143.

Based on Mark’s first step, what next step do you suggest? Explain why you suggest this step.</td></tr>
<tr><td colspan="2">Scoring Information</td></tr>
<tr><td>Key</td><td>Student suggests a valid next step with valid support.

Examples of valid steps and possible supports could be:
• Subtracting 43 from 48 so that the combination of Mark's step and this step makes an expression equivalent to the original expression.
• 143 – 48 is the same as 143 take away 43 and then take away 5 more. Mark found the first part is 100, so to finish you have to take away 5 more, to get 95.</td></tr>
</table>

The practice of collaborative mathematics can also be assessed through a scenario-based task. Illustration 3.24 presents the outline of a collaborative task, built from a classroom-based situation that could also involve mathematical modeling. Following the suggested student actions for each item in the task, a component of collaborative problem solving is given (see Illustration 3.20 for student performance associated with the collaborative skills of negotiating ideas and regulating problem solving listed in Hao et al. [2016]). The mathematical content of this grade 4 task focuses on fair sharing, a common elementary mathematics activity. Similar classroom activities can provide a solid foundation from which a collaborative mathematics task can be built.

Illustration 3.24. Collaborative Mathematics and Mathematical Modeling Example: Outline of a Scenario-Based Task Situated in a Classroom Setting

Start with a video clip or an avatar that sets up the task. Teacher shows students a transparent bin containing same-sized cubes. The bin cannot be opened by the students, but they can see that the cubes are different colors. Students pose questions about the cubes in the bin: How many cubes are there? How many cubes of each color are there? What will the cubes be used for? If we share the cubes, should everyone get the same number? What if there are cubes left over? Students begin to converge around questions involving the number of cubes in the bin, and about ways of determining the number of cubes in the bin. As part of the collaborative process, the assessed student is included as one of the students in the classroom.

Ideas for item content, below, reflect components of collaborative problem solving (Hao et al., 2016). Additional video or avatar communications should be provided between items.

1. The assessed student is asked to generate statements that define the goal of the task. (Student identifies the goal of the conversation.)
2. The assessed student is asked to provide a constructive response to a question about distributing cubes to groups. (Student suggests the next step for the group to take.)
3. After a process for distributing cubes to groups is determined, the assessed student is asked to describe what else the team needs to do. (Student expresses what is missing in the teamwork to solve the problem.)
4. When a classmate expresses frustration with the next step in the process of determining the number of cubes, the assessed student is asked to revoice the process to assist the classmate in understanding the process. (Student rephrases/completes the teammate's statement.)
5. The assessed student is asked to provide a representation for the number of cubes in the bin. When a classmate asks the student how that representation was determined, the assessed student provides support for the response by summarizing decision-making throughout the task. (Student reflects on what the group did.)

Developing items and tasks that involve the NAEP Mathematical Practice of Collaborative Mathematics can be challenging and time consuming, but the challenges may be lessened through collaborative item development. Additionally, existing resources may provide inspiration for task development. For example, the task in Illustration 3.24 was adapted from a sharing task used in an elementary classroom (Wickstrom & Aytes, 2018).

Balance of Mathematical Practices

The target percentage ranges of items for each NAEP Mathematical Practice are given in Exhibit 3.24. Most NAEP Mathematics Assessment items will feature one of the five NAEP Mathematical Practices (55 to 85 percent). The range of 55 to 85 percent allows flexibility in assessment and item development across grades 4, 8, and 12, while also ensuring that the majority of the assessment is designed to capture information on students' knowledge while they engage in NAEP Mathematical Practices. All NAEP Mathematical Practices will be represented in all grades and at least at the minimal levels. The relative emphasis on justifying and proving is based on its centrality across a range of mathematical activity; for example, the SBAC assessment targets justifying across multiple content categories, including modeling and data analysis, and communicating reasoning at every grade level.

Exhibit 3.24. Percentage Distribution of Items by NAEP Mathematical Practice

NAEP Mathematical Practice Area	Percentage of Items
Representing	10–15
Abstracting and Generalizing	10–15
Justifying and Proving	15–25
Mathematical Modeling	10–15
Collaborative Mathematics	10–15
Other	15–45

The remaining balance of items (15 to 45 percent) fall into the “Other” category and will assess knowledge of content without the item being designed to also assess a particular NAEP Mathematical Practice. Examples might include items that emphasize mathematical facts or procedural fluency or items that target practices that are not included in the five identified for the NAEP Mathematics Assessment. As noted earlier in this chapter, this could also include items that focus on algorithms, precision, or tool use.

Challenges

Together, the past several decades of research on mathematics thinking and learning and the consensus judgment of experts in mathematics education provide strong warrants for incorporating mathematical practices into the NAEP Mathematics Assessment. Despite widespread consensus on their importance, there are many challenges to assessing the NAEP Mathematical Practices. One is the interrelated nature of mathematical practices. Second, there is not consensus on how to define, let alone assess, mathematical practices. Finally, given the state of research and item development, it will be challenging to have sufficient numbers of items that assess student achievement with each NAEP Mathematical Practice, presenting challenges to reporting results on the Practices.

Although these challenges are formidable, they are not insurmountable. Existing state assessment programs include mathematical practices in their assessments. PISA has also been assessing mathematical practices for some time. Challenges can be addressed as the mathematical practices are incorporated into the 2026 NAEP Mathematics Assessment and refined over successive administrations. In addition, a special study to examine ways to report on mathematical practices to the general public is described in Appendix E. Despite these challenges, NAEP is clearly advancing mathematical practices as a core component of student achievement in mathematics, with the opportunity to become a leader in designing valid ways to assess the practices and report the results.

Exhibit 3.25A. Practices and Content Illustrations—Grade 4

In each cell, practice descriptors are included for a particular content area. The entries in this table are intended to be illustrative, not comprehensive. Included with some of the descriptors is italicized text providing the location of an item that is reflective of the descriptor.

Representing Grade 4				
Number Properties and Operations	**Measurement**	**Geometry**	**Data Analysis, Statistics, and Probability**	**Algebra**
Represent numbers or operations using visual models (e.g., base 10, number lines, fraction strips). *Illustration 3.1* *Illustration 4.13b* Recognize, translate between, interpret, and compare written, numerical, and visual representations of large numbers (e.g., thousands).	Select appropriate units related to representing or measuring an attribute of an object. Create visual representation of measurements or relationships between measurements.	Draw or sketch figures from a written description. Represent or describe figures from different views. Use a geometric model of a situation to draw conclusions.	Create a visual graphical, or tabular representation of a given data set. Compare and contrast different visual and graphical representations of a univariate distribution.	Recognize, describe, or extend numerical and geometric patterns using tables, graphs, words, or symbols. Translate between different representations of numerical expressions using symbols, tables, diagrams, or written descriptions.

Exhibit 3.25A. Practices and Content Illustrations—Grade 4 (continued)

Abstracting and Generalizing Grade 4				
Number Properties and Operations	**Measurement**	**Geometry**	**Data Analysis, Statistics, and Probability**	**Algebra**
Identify patterns in numbers or figures and generalize patterns in written or pictorial forms. Describe or extend a pattern or relationship to a larger set of numbers. Find structural relationships among sets of numbers. Generalize understanding of place value.	Make generalizations about areas of squares or rectangles. Extend quantified attributes to a larger set.	Generalize geometric properties by making connections across different figures and families of figures (e.g., triangles, quadrilaterals, polygons, polyhedra). Extend a geometric relationship from one or more figures to a family of figures.	Interpret graphical or tabular representations of data in terms of generalized phenomena (e.g., middle or median, range, mode, or shape). Make general conclusions about graphs of single sets of data (e.g., pictographs, bar graphs, dot plots).	Generalize a pattern appearing in a sequence or table, using words or symbols. *Illustration 3.7* Given a description, extend a pattern or sequence.

Exhibit 3.25A. Practices and Content Illustrations—Grade 4 (continued)

Justifying and Proving Grade 4				
Number Properties and Operations	**Measurement**	**Geometry**	**Data Analysis, Statistics, and Probability**	**Algebra**
Defend or counter claims about why a numerical relationship or pattern is valid or will always hold. *Illustration 3.22* Evaluate the appropriateness of an argument provided about properties or operations.	Defend or counter a claim about physical attributes, comparisons, or measurement properties. Choose a counterexample that disproves a claim about properties such as area, length, or volume.	Validate geometric conjectures (e.g., distinguish which objects in a collection satisfy a given geometric property and defend choices).	Evaluate the characteristics of a good survey and justify a survey's validity. Defend or counter conjectures offered based on a data set.	Make and justify conclusions and generalizations about numerical relationships. Given a pattern or sequence, construct, explain, or justify a rule to generate the terms of the pattern or sequence.

Exhibit 3.25A. Practices and Content Illustrations—Grade 4 (continued)

Mathematical Modeling Grade 4				
Number Properties and Operations	**Measurement**	**Geometry**	**Data Analysis, Statistics, and Probability**	**Algebra**
Use physical or virtual materials to build a model of a number pattern or to predict or estimate results of a continued pattern. Select and defend an appropriate method of estimation as a model for an estimation problem. Select appropriate properties or operations that can be used to build a model of a situation or solve a problem.	Identify the attribute(s) appropriate to measure in a given situation. Mathematize a contextual measurement situation to lead to a solution.	Use existing geometric models to solve mathematical or real-world problems.	Identify a statistical question to investigate in a given, open-ended or data-rich situation.	Identify a mathematical problem from a given situation that could be modeled numerically. Identify the variables needed to create an algebraic model of a situation.

Exhibit 3.25A. Practices and Content Illustrations—Grade 4 (continued)

Collaborative Mathematics Grade 4				
Number Properties and Operations	**Measurement**	**Geometry**	**Data Analysis, Statistics, and Probability**	**Algebra**
Add to a numerical model provided by others to complete a mathematical task. Evaluate others' interpretations of numbers from real-life contexts. Analyze the effect of another's estimation method on the accuracy of results.	Evaluate the validity of a measurement claim posed by others. Analyze others' solutions and suggest a critique of their solutions in a situation involving measurement. Attend to and make sense of the mathematical contributions of others in a situation involving measurement (e.g., revoice the work of others to clarify meaning of choice of measurement units).	Express and justify agreement or disagreement with a claim made by others in a geometric problem situation. Build on the work of others to geometrically model a situation.	Recognize and critique misleading arguments from data (e.g., from media or other people).	Verify the conclusions of others using algebraic/numerical properties.

Exhibit 3.25B. Practices and Content Illustrations—Grade 8

In each cell, practice descriptors are included for a particular content area. The entries in this table are intended to be illustrative, not comprehensive. Included with some of the descriptors is italicized text providing the location of an item that is reflective of the descriptor.

Representing Grade 8				
Number Properties and Operations	**Measurement**	**Geometry**	**Data Analysis, Statistics, and Probability**	**Algebra**
Represent word problems through visual models. Recognize, apply, create, or translate across multiple representations of fractions (e.g., visual models of equivalent fractions) and rational numbers (decimals, fractions, percents). *Illustration 4.9*	Select or use appropriate measurement instruments to determine the attributes of an object. Create visual representation of measurements or relationships between measurements.	Represent or describe figures from different views. Visualize and solve problems using geometry (e.g., using 2-D representations of 3-D objects). Use a geometric model of a situation to draw conclusions. Represent problem situations with geometric models to solve mathematical or real-world problems.	For a given set of data, create a visual, graphical, or tabular representation. *Illustration 4.12* Compare and contrast different visual and graphical representations of univariate and bivariate data. *Illustration 4.18a* Justify the use of a particular representation of data over another. Interpret visual representations to compare data sets, to draw inferences, or to make conclusions across two or more distinct data sets. Create and use scatterplots to represent the relationship between two variables and to estimate the strength of the relationship (strong, weak, none).	Use or create a graphical representation of a situation to draw conclusions. Translate between different representations of expressions using symbols, graphs, tables, diagrams, or written descriptions. *Illustration 3.3*

Exhibit 3.25B. Practices and Content Illustrations—Grade 8 (continued)

<table>
<tr><th colspan="5">Abstracting and Generalizing
Grade 8</th></tr>
<tr><th>Number Properties and Operations</th><th>Measurement</th><th>Geometry</th><th>Data Analysis, Statistics, and Probability</th><th>Algebra</th></tr>
<tr><td>Determine an expression for a recursive pattern.

Generalize, describe, or compare numerical properties and operations across different domains.

Extend a pattern or relationship to a larger set of numbers.

Find and generate structural relationships among sets of numbers.

Generalize findings about rational and irrational numbers.</td><td>Extend quantified attributes to a larger set.

Make connections between representations of different measurement systems.</td><td>Describe the general effects of dilations, translations, and rotations for two-dimensional figures.

Identify common elements across different figures and families of figures (e.g., triangles, quadrilaterals, polygons, polyhedra).

Extend a geometric relationship from one or more figures to a family of figures.</td><td>Interpret graphical or tabular representations of data in terms of generalized phenomena (e.g., shape, center, spread, clusters).

Generalize trends in data to suggest interpretations or infer conclusions.</td><td>Generalize a pattern appearing in a sequence, table, or graph using words or symbols.

Develop general rules for translating functions and graphs.

Create connections across representations.</td></tr>
</table>

Exhibit 3.25B. Practices and Content Illustrations—Grade 8 (continued)

Justifying and Proving Grade 8				
Number Properties and Operations	**Measurement**	**Geometry**	**Data Analysis, Statistics, and Probability**	**Algebra**
Defend a claim about why a numerical relationship or pattern is valid or will always hold. Find a counterexample to refute a claim about number properties or operations. Evaluate the appropriateness of a provided argument about properties or operations.	Defend a claim about physical attributes, comparisons, or measurement properties. Evaluate the validity of a provided argument making use of measurement. Find a counterexample to disprove a claim about properties such as area, length, or volume.	Verify properties of rotations, reflections, or translations. Create, test, and validate geometric conjectures (e.g., distinguish which objects in a collection satisfy a given geometric definition and defend choices). Defend claims about similarity of two-dimensional figures. Analyze a provided argument about geometric attributes or relationships.	Evaluate the characteristics of a good survey or of a well-designed experiment and defend the validity of surveys or experiments. Offer counter arguments in relation to conjectures about bivariate data.	Develop a valid mathematical argument based on properties of slope and intercept for linear functions. Justify functional relationships across different representational forms, such as tables, equations, verbal descriptions, or graphs.

Exhibit 3.25B. Practices and Content Illustrations—Grade 8 (continued)

Mathematical Modeling Grade 8				
Number Properties and Operations	**Measurement**	**Geometry**	**Data Analysis, Statistics, and Probability**	**Algebra**
Build a model of a situation for an estimation problem. Communicate and defend a decision about a physical or virtual model involving number and/or operation to an audience for feedback.	Mathematize a contextual measurement situation to lead to a solution. Evaluate the reasonableness of a model unit for an attribute in a real context.	Visually model the effects of successive (or composite) transformations of figures in the plane. Construct geometric models using physical or virtual materials to solve mathematical or real-world problems.	Identify a statistical question to investigate in a given, open-ended or data-rich situation. Create or use a statistical model to answer a statistical question or make a prediction about a data set. Create or use a statistical model to assess the validity of a statistical claim.	Identify the variables needed to create an algebraic model of a situation. Write algebraic relationships, expressions, equations, or inequalities to model real-world situations.

Exhibit 3.25B. Practices and Content Illustrations—Grade 8 (continued)

<table>
<tr><th colspan="5">Collaborative Mathematics
Grade 8</th></tr>
<tr><th>Number Properties and Operations</th><th>Measurement</th><th>Geometry</th><th>Data Analysis, Statistics, and Probability</th><th>Algebra</th></tr>
<tr><td>Build on a numerical model provided by others to complete a mathematical task.

Analyze the effect of another's estimation method on the accuracy of results.

Reflect on the work of others to extend a numerical pattern.</td><td>Evaluate the validity of a measurement claim posed by others.

Engage in joint thinking to reach consensus about a measurement situation.

Analyze others' solutions and suggest a critique of their solutions in a situation involving measurement.</td><td>Express and justify agreement or disagreement with a claim made by others in a geometric problem situation.

Build on the work of others to geometrically model a situation.

Evaluate the merit of others' geometric ideas.

Connect across geometric ideas contributed by others in a problem-solving situation.</td><td>Choose a worthwhile statistical question from a set offered by others about a problem situation or context involving data.

Recognize and critique misleading arguments from data (e.g., from media or other people).

Revoice the work of others in addressing a statistical or probabilistic situation.

Analyze the models constructed by others to evaluate a new data set.</td><td>Verify the conclusions of others using algebraic properties.</td></tr>
</table>

Exhibit 3.25C. Practices and Content Illustrations—Grade 12

In each cell, practice descriptors are included for a particular content area. The entries in this table are intended to be illustrative, not comprehensive. Included with some of the descriptors is italicized text providing the location of an item that is reflective of the descriptor.

Representing Grade 12				
Number Properties and Operations	**Measurement**	**Geometry**	**Data Analysis, Statistics, and Probability**	**Algebra**
Create and justify solutions to word problems through numeric representations and operations. Represent, interpret, or compare expressions or problem situations involving absolute values.		Represent or describe figures from different views. Visualize and solve problems using geometry (e.g., using 2-D representations of 3-D objects). Represent problem situations with geometric models to solve mathematical or real-world problems.	For a given set of data, create a visual, graphical, or tabular representation of the data. Compare and contrast different visual and graphical representations of univariate and bivariate data. Interpret visual representations to compare data sets, to draw inferences, or to make conclusions across two or more distinct data sets. Create and use scatterplots to represent the relationship between two variables and to estimate the strength of the relationship (strong, weak, none).	Use or create a graphical representation of a situation to draw conclusions. *Illustration 4.18b* Translate between different representations of expressions using symbols, graphs, tables, diagrams, or written descriptions. Express linear and exponential sequences in recursive or explicit forms given a table.

Exhibit 3.25C. Practices and Content Illustrations—Grade 12 (continued)

Abstracting and Generalizing Grade 12				
Number Properties and Operations	**Measurement**	**Geometry**	**Data Analysis, Statistics, and Probability**	**Algebra**
Determine a generalized expression for a recursive pattern. Extend properties of numbers from one system to another (for instance, extend the properties of exponents to rational exponents). Generalize, describe, or compare numerical properties and operations across different domains or number systems. Extend a pattern or relationship to a larger set of numbers. Find and generate structural relationships among sets of numbers.	Generalize the effect of proportions and scaling for area and volume. Extend trigonometric formulas to determine triangle unknowns.	Generalize relationships such as congruence, similarity, or orientation between figures and their images under transformation. Extend a geometric relationship from one or more figures to a family of figures. Develop generalizations about transformations that preserve the area or volume of figures.	Interpret graphical or tabular representations of data in terms of generalized phenomena (e.g., shape, center, spread, clusters). Organize and display data in order to recognize and make inferences from patterns in the data. Notice patterns of outcomes in a probability situation. Generalize trends in data to suggest interpretations or infer conclusions. Develop generalizations about how linear transformations of one-variable data affect mean, median, mode, range, interquartile range, and standard deviation.	Extend and generalize numerical patterns, including arithmetic and geometric progressions. *Illustration 3.6* Compare and generalize properties of linear, quadratic, rational, and exponential functions. Identify commonalities within and across function families. Develop general rules for translating functions and graphs. Create connections across representations.

Exhibit 3.25C. Practices and Content Illustrations—Grade 12 (continued)

Justifying and Proving Grade 12				
Number Properties and Operations	**Measurement**	**Geometry**	**Data Analysis, Statistics, and Probability**	**Algebra**
Find a counterexample to refute a claim about number properties or operations. Prove numerical relationships through developing deductive arguments, engaging in proof by exhaustion, or employing mathematical induction. Evaluate the validity of a provided argument about properties or operations. *Illustration 3.13* Analyze or interpret a proof by mathematical induction about the properties of numbers. *Exhibit 3.14* Justify relationships between properties of number systems, including natural numbers, integers, rational numbers, real numbers, and complex numbers.	Justify or prove a claim about physical attributes, comparisons, or measurement properties. Explain why a given attribute can be appropriately measured by the chosen quantity and unit. Evaluate the validity of a provided argument making use of measurement. Find a counterexample to disprove a claim about properties such as area, length, or volume. Prove conjectures about trigonometric identities.	Justify relationships of congruence and similarity; apply these relationships using scaling and proportional reasoning. *Exhibit 3.11* Create, test, and validate geometric conjectures (e.g., distinguish which objects in a collection satisfy a given definition and defend choices). Analyze a provided argument about geometric attributes or relationships. Use given definitions and theorems to prove geometric conjectures. Develop justifications and proofs that rely on a variety of representational modes (e.g., two-column, paragraph). Discuss the implications that a definition of a type of figure has on the figure properties.	Critique the validity of surveys or experiments. Justify or prove conjectures about probability. Create and explore counting arguments in order to develop and justify conjectures.	Create, validate, and justify conclusions and generalizations about functional relationships. Verify a conclusion using algebraic properties. Prove algebraic relationships through developing deductive arguments, finding counterexamples, engaging in proof by exhaustion, and employing mathematical induction. *Exhibit 3.12*

Exhibit 3.25C. Practices and Content Illustrations—Grade 12 (continued)

Mathematical Modeling Grade 12				
Number Properties and Operations	**Measurement**	**Geometry**	**Data Analysis, Statistics, and Probability**	**Algebra**
Select appropriate properties or operations that can be used to build a model of a situation or solve a problem. Create a physical or virtual model involving number and/or operation.	Select or use a model unit for an attribute to be measured and defend the use of that unit. Mathematize a contextual measurement situation to lead to a solution. Create a model to convert between two measurement systems. Construct scale drawings to be used as measurement models of objects in problem situations.	Create a geometric model of a physical object. Discuss differences in solutions caused by having used a simplified model. Use existing geometric models to solve mathematical or real-world problems. Visually model the effects of successive (or composite) transformations of figures in the plane. Construct geometric models using physical or virtual materials to solve mathematical or real-world problems. Predict the results of combining, subdividing, and transforming geometric figures.	Identify a statistical question to investigate in a given, open-ended or data-rich situation. Use a statistical model to answer a statistical question or make a prediction about a data set. Create a probability model to calculate or estimate the probability of an event. Compare and contrast theoretical probabilities with results from experimental probabilities in a simulation.	Identify a mathematical problem from a given situation that could be modeled algebraically. Identify the variables needed to create an algebraic model of a situation. Write algebraic relationships, expressions, equations, or inequalities to model real-world situations. Revise an existing algebraic model based on introducing new variables or parameters. Build or apply a mathematical model of a financial situation (e.g., a monthly family budget, or a car loan).

Exhibit 3.25C. Practices and Content Illustrations—Grade 12 (continued)

Collaborative Mathematics Grade 12				
Number Properties and Operations	**Measurement**	**Geometry**	**Data Analysis, Statistics, and Probability**	**Algebra**
Build on a numerical model provided by others to complete a mathematical task. Analyze the effect of another's estimation method on the accuracy of results. Reflect on the work of others to extend a numerical pattern. Evaluate the mathematical reasonableness of a peer's mathematical contribution.	Evaluate the validity of a measurement claim posed by others.	Express and justify agreement or disagreement with a claim made by others in a geometric problem situation. Attend to the contributions of others in collaboratively generating a geometric proof. Build on the work of others to geometrically model a situation. Generalize across geometric ideas contributed by others in a problem-solving situation.	Revoice/restate the work of others in addressing a statistical or probabilistic situation. Analyze the models constructed by others to evaluate a new data set.	Verify the conclusions of others using algebraic properties.

OVERVIEW OF THE ASSESSMENT DESIGN

This chapter provides an overview of the major components of the mathematics assessment design, which includes the types of assessment tasks and item formats and how they can be used to expand the ways in which students are asked to demonstrate what they know and can do in mathematics. In addition, this chapter describes how the assessment is distributed across the five mathematics content areas described in Chapter 2 and the five NAEP Mathematical Practices in Chapter 3. The 2026 Framework intentionally emphasizes increased access for students—including English language learners and students with disabilities—to demonstrate their mathematics understanding. Scholarship has demonstrated that students of various ethnic, racial, economic, and cultural backgrounds have salient differences that matter to the format and design of assessment items for inclusiveness (Solano-Flores, 2011). In particular, the 2026 NAEP Mathematics Assessment will continue to use concepts of universal design for assessment to increase inclusiveness and assessment validity (Thompson, Johnstone, & Thurlow, 2002).

Previous NAEP Mathematics Assessments included only discrete items, which stand alone or comprise a composite item. Discrete items consist of selected response and constructed response item types. In order for students to demonstrate what they know and can do with respect to the range of mathematics content knowledge and NAEP Mathematical Practices in the Framework, the 2026 NAEP Mathematics Assessment includes a new item assessment format: scenario-based tasks. Scenario-based tasks have both context and extended storylines to provide opportunities to demonstrate facility with the integrated nature of mathematics content knowledge and NAEP Mathematical Practices.

Two fundamental aims motivate the expansion. First, there is a need to ground the NAEP assessment in relevant tasks and familiar contexts to provide a better measure of student content knowledge and mathematical practices (Eklöf, 2010). Second, by expanding item types and thoughtfully using technology, the NAEP Mathematics Assessment continues to provide greater access to all students, diversifies the ways in which student achievement can be recognized and measured, and more robustly assesses both what students know and what they can do. For example, graphics can be presented in color with greater clarity and with a tool to zoom in and out (Sireci & Zenisky, 2006).

Technology provides opportunities for assessment, but with each opportunity come myriad constraints and repercussions that must be considered. For example, introducing a new format for items on the NAEP Mathematics Assessment that is interactive or discussion-based requires that great care be taken to ensure that the design is accessible to students, that students have ample time to understand how to engage with the item, and that students have had opportunities to experience the task type. Familiarity with digital technology in general, and with specific digital tools in particular, can influence student performance (Dunham & Hennessy, 2008). Other potential threats to assessment validity are the accessibility of tools and the affordances for students with and without certain disabilities. Due to differential access to, use of, and outcomes stemming from student experiences with technologies in and out of school (Warschauer & Matuchniak, 2010), development work should address known and potential implementation challenges and identify ways to mitigate issues of access in doing the assessment that could occur in under-resourced communities (Warschauer, 2016). A goal of the NAEP Mathematics Assessment is not to disadvantage students by virtue of the assessment's technology.

Item Development

Chapter 2 describes, for each grade level, the content objectives in each of five areas of mathematics: Number Properties and Operations; Measurement; Geometry; Data Analysis, Statistics, and Probability; and Algebra. Chapter 3 describes the five NAEP Mathematical Practices that are the targets for assessing mathematical activity across all grade levels: Representing, Abstracting and Generalizing, Justifying and Proving, Mathematical Modeling, and Collaborative Mathematics. Those chapters, combined with the guidelines in this chapter, focus on realizing the intent of the Framework in developing items used on the assessment.

The guidelines offered here highlight only some of the critical considerations in item development, concentrating on topics specific to the NAEP Mathematics Assessment. Item writers should refer to directions for developing items provided by the Governing Board and its designees in addition to the information in this document.

Item Characteristics

The specific components of an item are determined by the item format. Two components are constant across all item formats: (1) the item stem and (2) the response. The item stem, also known as the stimulus, is the introduction to the item and the question asked of, or directive given to, students. The item stem should provide all of the necessary information for students to respond, clearly laying out for the students what is being asked and the expected response method. The response method is determined by the item format.

Illustration 4.1 is a multiple choice item with the main item components labeled. Note that the rationales – the support for the inclusion of the response options as correct or plausible – are used during item development and item review, but are **not** part of a student-facing NAEP item.

Illustration 4.1. Components of a Multiple Choice Item

The expanded form of a number is shown.

$$8\times10{,}000 + 4\times100 + 6\times10$$

What is the number in standard form?

Item Stem

A. 846
B. 8,460
C. 80,460
D. 84,600

Response Options

Rationales for response options:

Rationale A: Uses the digits 8, 4, and 6 in order of appearance, but not with associated place value.
Rationale B: Uses the digits 8, 4, and 6 in order of appearance, but writes 8 as the digit in the thousands place.
Rationale C: Correct. 80,000 + 400 + 60 = 80,460.
Rationale D: Uses the digits 8, 4, and 6 in order of appearance, but writes 4 as the digit in the thousands place and 6 as the digit in the hundreds place.

General Principles of Item Writing

NAEP items will be developed in accordance with recommended practice and the Governing Board Item Development and Review Policy (2002). The Board's policy includes principles about item writing that apply to all NAEP assessments.

Scenario-Based Tasks

The 2026 NAEP Mathematics Assessment will include existing and new discrete items as well as scenario-based tasks. The goal of scenario-based tasks is to provide evidence of students' ways of knowing and doing mathematics. Current and future NAEP Mathematics Assessments can take advantage of evolving digital technologies to create the next generation of scenario-based tasks, as well as yet-to-be-imagined items and tasks. Other NAEP frameworks have set a foundation for scenario-based tasks. For example, since 2009 the NAEP Science Framework has called for the use of interactive computer tasks, and the NAEP Technology and Engineering Literacy (TEL) Framework has done so since its start in 2014 (Governing Board, 2014c, 2014d). Examples of scenario-based TEL tasks can be found online (Governing Board, 2014e).

The defining features of the scenario-based task for the 2026 NAEP Mathematics Assessment are an authentic context, in which students can imagine themselves, with a motivating question or goal, along with item design that supports exploration. The motivating goal for a scenario-based task might be to solve a particular problem or to complete a certain mission within the scenario. The goal provides the driving rationale for the tasks that the student will perform. It offers a storyline that helps build needed background, defines the task's relevance and coherence, and motivates the student to engage with the scenario-based task.

Within one scenario-based task, a student may complete multiple items that vary in format, with both constructed and selected response item types (details are provided in the Item Types section). Within a scenario-based task, each item is in some way related to, or builds on, the next item as part of the cohesive experience. Such tasks may be well suited to addressing the intersecting nature of the mathematics content and the NAEP Mathematical Practices illustrated in Exhibits 3.25A–3.25C at the end of Chapter 3. Scenario-based tasks may also be especially well suited to measuring the highly iterative or interactional nature of the NAEP Mathematical Practices described in Chapter 3.

An advantage of digital delivery of the assessment is that scenario-based tasks can use multimedia (e.g., images, video, and animation, in addition to future technologies) to present the settings for the assessment items. As a result, non-mathematical linguistic demand might be reduced while mathematical rigor is maintained. Multimedia can also better scaffold the background understanding that examinees may need to complete a given item. For example, video segments or animations that a student observes, along with text, numbers, and graphics, can convey information necessary for the task to be accomplished. In developing such scenario-based tasks, related design decisions should serve a particular purpose and not be extraneous or presented simply for visual interest. While in many cases relevant multimedia content can have a positive impact on student engagement and performance, it is also possible that it may introduce competition of attention between visual and auditory channels (Fawcett, Risko, & Kingstone, 2015). When multimedia content is included in a scenario-based task, developers need to ensure that the multimedia content is used productively and minimizes such competition.

Within a scenario-based task, students are given opportunities to select tools from a toolkit and use them to solve problems. For example, students might be asked to select a graphing or spreadsheet tool or to use a simulation. Various digital and physical tools may be made available, depending on the scenario. These might take the form of chat/texting, or presentation tools for communication tasks, if deemed relevant to the mathematical understanding being assessed.

When designing tools for a scenario-based task, it is necessary to determine which elements of a tool are needed for the activities in the scenario and which features are used by students. For example, only those functions of a spreadsheet tool that are directly relevant to a given item might be provided. It is not necessary to provide all of the other features of the spreadsheet tool. In fact, including every feature could be distracting to students and could produce measurement error. Additionally, students are not expected to know how to use all tools in a scenario-based task prior to starting the task. In these cases, instructions and practice using the tool are embedded in the task before the tool is needed or used to complete the task.

An important consideration for assessment developers when designing scenario-based tasks is to ask what is gained through the selection of a scenario as assessment context. A robust scenario will allow examinees to interact with task components in multiple ways, explore alternative outcomes and explanations, find multiple solution paths, and demonstrate their thinking. Students could also evaluate the outcomes of the choices they make and convey their understanding of mathematical concepts in diverse ways. For example, one scenario-based task may engage students in a range of mathematical practices and foreground one content area.

Interactive scenario-based tasks can elicit rich data, providing evidence of NAEP Mathematical Practices that are difficult to measure with more conventional items and tasks. For example, measuring collaboration has long been a challenge in assessment. Novel methodological approaches have explored discipline-specific student collaborative activity through the use of performance outcomes and process data from scenario- and simulation-based collaborative assessment (Andrews et al., 2017). These approaches can be used to better assess the NAEP Mathematical Practice of Collaborative Mathematics.

As illustrated in the PISA example in Chapter 3 (see Exhibits 3.20–3.21), validated scenario-based tasks that assess collaborative problem solving already exist. In that example, the task was structured as a dialogue with a collaborative team made up of avatars and assessed students in a way that is nearly impossible to do using only discrete item sets. In contrast, Exhibit 4.1 (based on a grade 8 *Stacking Chairs* task from the Silicon Valley Mathematics Initiative [2016]) illustrates a set of discrete items that are scenario-based, presented in a non-digital environment. Notably lacking from this example are supporting multimedia and tools.

Exhibit 4.1. Grade 8 Scenario Example

You, Lee, and Pat are the team organizing the spring concert at your school. The school has a large room with a stage but the team will need to arrange for renting chairs from a local company. The chairs must be put in a storage room before the concert. The chairs can be stacked. The team stacked some chairs and measured the heights of the stacks. Below are the notes the team made.

The height of stacked chairs

5 chairs are 51 inches high

3 chairs are 45 inches high

8 chairs are 60 inches high

1. How tall are two chairs stacked together? ________ inches

Lee suggests the chairs be stacked in groups of 10.

2. How tall is a stack of 10 chairs? ________ inches
   Show how you figured it out.

The team decides that groups of 10 chairs will take up too much floor space. The team wants an equation to know how tall a stack will be if you know the number of chairs.

3. Write an equation to find the height, *y*, if the number of chairs in a stack is *x*.

4. Explain how Pat can use the equation you wrote to determine the height of 28 chairs.

The storage room is 15 feet tall. Three feet of space above the stack of chairs is needed (to take chairs off the stack).

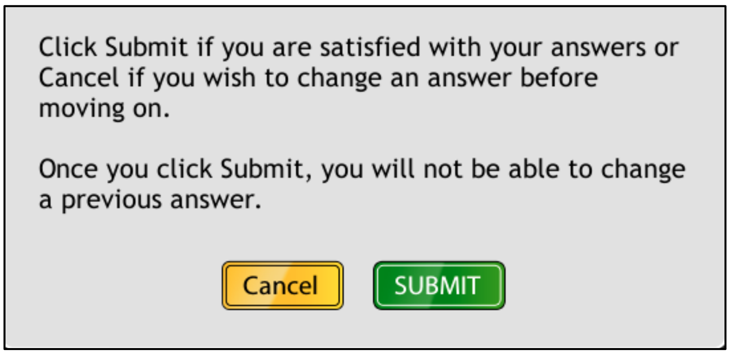
5. How many chairs can be in a stack and still fit in the storage room? ______ chairs
   Show how you figured it out.

6. There will be 200 chairs for the audience. What else would the team need to know in order to determine whether or not all 200 chairs will fit in the storage room? Why is the information needed?

Note that the response to item 4 in Exhibit 4.1 is dependent on the response to item 3. When dependencies such as this occur, scoring needs to account for a correct answer to item 4 based on an incorrect answer to item 3. On a digitally based assessment, the task can be presented in a way that removes the dependency. In administering the item, students could be asked to review and submit their answer to item 3 before accessing item 4. The revised digital version of item 3 from Exhibit 4.1 could read:

3. The number of chairs in a stack is represented by x. Write an equation to determine the total height, y, in inches, of the stack of chairs.

Upon completion of the item, the student is notified that once the answer is submitted, it cannot be changed. The image below shows text displayed to students during administration of TEL tasks, which can be adapted for use on the mathematics assessment.



To allow for completion of item 4 without reference to a response provided for item 3, item 4 could be revised to give an equation that represents the height of a stack of x chairs and ask students to use the equation to determine the height of a stack of 28 chairs. For example, item 4 could be revised to read:

4. Lee writes the equation shown to determine the total height, y, in inches, of a stack of x chairs.

$$y = 36 + 3x$$

Explain how to use the equation that Lee wrote to determine the height, in inches, of a stack of 28 chairs. As part of your response, determine the height, in inches, of a stack of 28 chairs.

For additional examples that avoid dependencies between related item parts, see the TEL scenario-based examples (e.g., the *Andromeda Task* [Governing Board, 2014a]).

A richer version of the *Stacking Chairs* task, as a scenario-based task, is provided in Illustration 4.2. The context puts the students in the task as part of a team determining whether chairs can be stacked in a storage room. Item text in Parts B, C, and D presents content differentiation for grade 8 and grade 12. Included with these versions of the task are development notes and scoring information. Additional information on scoring is provided later in this chapter.

Illustration 4.2. Alternative *Stacking Chairs* Task

You, Chi, and Alma are the team organizing seating for the spring concert at your school. The audience will sit in chairs that must be put in a storage room after the concert. The team needs to determine whether all 200 chairs can be stored in the room.

The chairs are identical and can be stacked. The team stacked some chairs and measured the heights of the stacks. Below are the heights the team measured.

The height of stacked chairs

5 chairs are 51 inches high

3 chairs are 45 inches high

8 chairs are 60 inches high

Part A.

What is the height, in inches, of a stack of 2 chairs?
[Correct response: 42 (inches)]

Part B (Grade 8).

The team wants a way to determine the height of a stack of chairs when the number of chairs in the stack is known. Write an equation that can be used to determine h, the total height, in inches, of a stack of n chairs.

On Screen: Click Submit if you are done with your answer or cancel if you wish to change your answer before moving on. Once you click Submit, you cannot change the answer.

[Scoring Information: Student response should be equivalent to $h = 3n + 36$.*]*

Part B (Grade 12).

The team wants a way to determine the height of a stack of chairs when the number of chairs in the stack is known. The team will stack chairs on a cart. The cart adds 18 inches to the total height of a stack.

Write an equation that can be used to determine h, the total height, in inches, from the ground to the top of n chairs stacked on a cart. Explain how you determined your equation.

On Screen: Click Submit if you are done with your answer or cancel if you wish to change your answer before moving on. Once you click Submit, you cannot change the answer.

[Scoring Information: Student response should mathematically support an equation equivalent to $h = 3n + 54$.
Sample student response: Since the height of a stack of 5 *chairs is* 51 *inches and the height of a stack of* 3 *chairs is* 45 *inches, each additional chair increases the total height of the stack by* 3 *inches. Since* $45 – 6 = 39$*, the height of one chair is* 39 *inches. So, the height, h, of a stack of n chairs would be* $39 + 3(n – 1)$ *or* $3n + 36$. *Since the height of the cart adds* 18 *inches to the total height, the height of a stack of n chairs on a cart would be* $h = 3n + 54$.*]*

Part C (Grade 8).

Chi writes the equation shown to determine *h,* the total height, in inches, of *n* chairs stacked on a cart.

$$h = 36 + 3n$$

Explain how to use the equation that Chi wrote to determine the height, in inches, of a stack of 28 chairs. As part of your response, determine the height, in inches, of a stack of 28 chairs.

On Screen: Click Submit if you are done with your answer or cancel if you wish to change your answer before moving on. Once you click Submit, you cannot change the answer.

[Scoring information: Student response should mathematically support a height of 120 *inches.*
Sample student response: Since n *represents the number of chairs, substitute* 28 *for n. Multiply* 28 *by* 3. *Then add* 36. *The height of a stack of* 28 *chairs is* 120 *inches.]*

Part C (Grade 12).

Alma writes the equation shown to determine *h,* the total height, in inches, of a stack of *n* chairs.

$$h = 54 + 3n$$

After the chairs are stacked on the carts, they will be stored in a room that is 12 feet high. A space of 3 feet is needed above the top of each stack of chairs so that chairs can be taken off the cart.

The team has determined that no more than 10 carts can be put into the storage room. Using Alma's equation, determine whether or not all 200 chairs can be stacked on carts and stored in the room. Show your work or explain how you determined your answer.

[Scoring information: Student response should mathematically support that all 200 *chairs cannot be stored in the room. Correct response may or may not include reference to shorter doorway and vertical fit through doorway rather than vertical fit in the room itself.*
Sample student response: Since 10 *carts will fit in the storage room, when each cart has the same number of chairs stacked on it, each cart will have* 20 *chairs stacked on it:* $200 \div 10 = 20$. *Using the equation* $h = 3n + 54$*, the height of each stack will be* 114 *inches:* $3 \times 20 + 54 = 60 + 54 = 114$. *Since* 3 *feet are needed at the top of each stack, the total height needed for each stack is* $114 + 36 = 150$ *inches. The height of the room is* 12 *feet, which is* 144 *inches. Since* 150 *inches are needed for each stack, all* 200 *chairs cannot be stored in the room.]*

Part D (Grade 8).

The team will put 200 stacking chairs into the storage room. What other information does the team need to know to determine whether all 200 chairs will fit in the storage room? Why is the information needed?

[Scoring information: Student response should include information about floor space in the storage room, other dimensions of the doorway (e.g., width of opening) and other dimensions of the chair stack related to width and length of the stack (in addition to the height information). Justification might include a need for the stack to fit through the doorway of the storage room and establishing lower bounds on fit for both the doorway and the room.]

Item Development Information

Part A (Grades 8 and 12).
Objective Alignment: Algebra, 1.a
NAEP Mathematical Practice Alignment: None

This item serves as a lead-in to the task. Although the relationship given by the heights of the chairs is linear, students may not use a linear relationship to determine the height of one chair. However, students will need to use the difference of 3 inches between the height of a stack of n chairs and the height of a stack of $n + 1$ chairs, focusing on the application of a determined pattern to answer the question asked.

Part B (Grades 8 and 12).
Objective Alignment: Algebra, 3.b
NAEP Mathematical Practice Alignment: Abstracting and Generalizing; Representing

Although students could use the height from Part A to determine the equation, they do not need to. Instead, a student could use the difference of 3 inches between the height of a stack of n chairs and the height of a stack of $n + 1$ chairs to determine that 15 inches of a stack of 5 chairs are the seats. Since the total height is 51 inches, 36 inches are constant.

Part C (Grade 8).
Objective Alignment: Algebra, 4.a
NAEP Mathematical Practice Alignment: None

Part C presents a correct equation and asks students to determine the height of a stack containing a specified number of chairs. This item is intended as a scaffold to the open-ended item in Part D. In Part C, students need to provide an explanation for how the equation could be used to determine the height of a stack of chairs. The height of one or more stacks of chairs is a component in the process of determining whether or not all of the chairs will fit in the storage room, which is the focus of Part D.

Part C (Grade 12).
Objective Alignment: Algebra, 4.c
NAEP Mathematical Practice Alignment: Justifying and Proving

Part C extends the thinking done in Part B by requiring students to use an equation to determine whether additional constraints can be met when placing the carts in a storage room. Students might approach this item by starting with the height of the room or the height of a stack of chairs.

Part D (Grade 8)
Objective Alignment: Algebra, 4.c
NAEP Mathematical Practice Alignment: Mathematical Modeling

Part D provides an opportunity for students to consider constraints and limitations to putting the chairs in the storage room. The open-ended nature of this question increases complexity while also allowing for the consideration of multiple measurements that impact the storage of the chairs in the room. The request for constraints and limitations applies in general to the locating of chairs in the room and associates this item to component (b) of the mathematical modeling cycle since the responses serve to identify additional information needed to complete the task.

One of the affordances of scenario-based tasks is in the ability to leverage digital tools to make the content, and thereby the evidence produced, more accessible and authentic. For this particular task, the inclusion of a virtual measuring tape could allow students to measure heights of stacks of varying numbers of chairs. The measuring could be within a scale drawing context, or could allow for realistic measurements within a virtual environment. In either case, the student would need to line the measuring tool up properly to measure the height. To assist with this, the digital environment could have the measuring tape click in place (visually and audibly) so that a more accurate measurement could be made.

Due to their capacity to replicate authentic situations (i.e., experiences that students may encounter in their lives), scenario-based tasks have the potential to provide a level of accessibility and support for student engagement with the assessment that other types of assessment tasks do not. Additionally, scenario-based tasks provide opportunities to simultaneously assess multiple practices or content areas. However, a block of scenario-based tasks may provide less measurement information than a block of discrete items in the same amount of assessment time; scenario-based tasks typically require a longer duration to reach optimal reliability (Jodoin, 2003).

Scenario-based tasks will take students about 10–20 minutes to complete. Longer scenario-based tasks may include a greater number of embedded assessment requirements and items to which a student is asked to respond. The discussion of the balance of item types later in this chapter provides a general range to allow item developers greater flexibility to fulfill assessment design blocks.

Leveraging Existing NAEP Items to Create Scenario-Based Tasks

All of the general principles for item writing discussed in this document apply to the development of scenario-based tasks. However, the development of a well-written scenario-based task is not easy. The authors of the 2019 Trends in International Mathematics and Science Study (TIMSS) Framework noted that TIMSS problem solving and inquiry tasks (PSIs), which have characteristics similar to NAEP scenario-based tasks, were challenging and time consuming to build (Mullis & Martin, 2016). Therefore, to aid in task development for NAEP mathematics, some illustrated suggestions are offered, building on existing NAEP TEL specifications and on existing NAEP items as starting points.

The 2014 TEL specifications (Governing Board, 2014d) suggested use of a *scenario shell* to help think through the components of a task, including the problem to be solved and the practices and objectives being assessed. An example from the *TEL Assessment and Item Specifications* is shown in Illustration 4.3a. An adaptation for NAEP mathematics item development is shown in Illustration 4.3b.

Illustration 4.3a. NAEP TEL Sample Scenario Shell

Grade	4, 8, or 12
Major Assessment Area(s)	Technology and Society Design and Systems Information and Communication Technology
Context	*What is the context of the scenario?*
Problem	*What are the big ideas for the students?*
Available Resources and Information	*What is given to the student to solve the problem?*
Tools Used	*What domain-specific tools (virtual and actual) will the students use?*
Practices	*Which of the NAEP practices will be addressed?*
Assessment Targets	*Which of the NAEP targets will be addressed?*

Illustration 4.3b. NAEP Mathematics Sample Scenario Shell

Grade	4, 8, or 12
Major Content Area(s)	Number Properties and Operations Measurement Geometry Data Analysis, Statistics, and Probability Algebra
Context	*What is the context of the scenario?*
Problem	*What are the big ideas for the students?*
Available Resources and Information	*What is given to the student to solve the problem?*
Tools Used	*What domain-specific tools (virtual and actual) will the students use?*
NAEP Mathematical Practice(s)	*Which of the NAEP Mathematical Practices will be measured?*
NAEP Mathematics Objective(s)	*Which of the NAEP content objectives will be measured?*

Bicycle Trip Example: Grade 8 Scenario-Based Task

Consider the *Bicycle Trip* item introduced in Chapter 3 (Exhibit 3.4) and included again in Illustration 4.4 for reference.

Illustration 4.4. NAEP *Bicycle Trip* Item

The graph above represents Marisa's riding speed throughout her 80-minute bicycle trip. Use the information in the graph to describe what could have happened on the trip, including her speed throughout the trip.

During the first 20 minutes, Marisa *[textbox]*

From 20 minutes to 60 minutes Marisa *[textbox]*

From 60 minutes to 80 minutes Marisa *[textbox]*

Using the NAEP Mathematics Sample Scenario Shell and the original *Bicycle Trip* item, an outline of a scenario-based task was developed (see Illustration 4.5). During this process, the context of the original item was revisited to consider topics of interest for eighth graders.

With the multimedia capabilities of online administration of scenario-based tasks, consideration was given to the unique opportunities for content presentation as a way to connect a version of the graphical representation from the original item to a different type of representation in the new task. The choice to use video clips as a mode of representation provides a level of engagement not offered by the original task.

Illustration 4.5. Grade 8 Scenario Shell Adaptation of *Marisa's Bicycle Trip*

Grade	8
Major Content Area(s)	Algebra
Context	Ordering video clips of a bicycle trip
Problem	Given a graph and a set of video clips, order the clips to show Marisa's bicycle trip.
Available Resources and Information	video clips graphical representation
Tools Used	interactive item component
NAEP Mathematical Practice(s)	Representing
NAEP Mathematics Objective(s)	Algebra 2.a – Translate between different representations of linear expressions using symbols, graphs, tables, diagrams, or written descriptions.

After the scenario shell was completed, an initial draft of a portion of the task was developed. This draft is shown in Illustration 4.6. Since revisions to the original graphic are likely needed and technology features will be applied, some italicized notes are included within the item to illustrate thinking about these item components. Additional item parts could be added to consider questions that can be answered about Marisa's trip or to apply the same relational thinking to a different context. As the multimedia features of the mathematics assessment are configured, this task could be developed further and could continue to be refined (e.g., using a satellite version of a map where the student can visually see the topography and then draw the route Marisa took on the map, based on the graph; as the student draws the route, there could be a timer or clock on the side that adjusts as the route is drawn; changing the original problem from time to distance, then including an odometer on the side).

To build from items in the existing item pool, a scenario-based task based on an unreleased NAEP item could include the original item as a part of the task. For example, based on the *Bicycle Trip* item, the original item might be used as Part A, to have students talk about the rate at which Marisa rides. The new content of the task, the ordering of the video clips, could be included as Part B.

Illustration 4.6. Draft Grade 8 Scenario-Based Task Adaptation of *Marisa's Bicycle Trip*

You are producing a video to tell the story of Marisa's bicycle trip. You will order four video clips. To assist you, an editor has created a graph showing the relationship between the number of minutes Marisa rode her bicycle and her speed.

[*Art note: Adjust the time shown on the axis depending on whether videos are meant to be real time (e.g., a 4-minute trip) or representations of segments of a longer trip.*]

Watch each video clip. Then, put the four clips in order so that they represent the graph of Marisa's bicycle trip.

[*Technology implementation note: Create a tab for each of the four clips. Label each tab "Video Clip <letter>", with <letter> replaced with A, B, C, and D. Create a fifth tab for ordering the clips to show the trip. Label the fifth tab "Order the Clips". Consider the potential to merge the ordered clips all together to show the trip in its entirety.*
Video Clip description and scoring order:
Clip A: shows Marisa riding at constant speed (order: second)
Clip B: shows Marisa stopped (order: fourth)
Clip C: shows Marisa riding at a decreasing speed (order: third)
Clip D: shows Marisa riding at an increasing speed (order: first)
Note that video clips should not give the actual speed at which Marisa is riding.]

Tab development:
The graph of Marisa's Bicycle Trip should be shown on each tab.

Text for use with Video Clips A, B, and C:
Watch the video clip. Then select the tab for the next video.
[*Include play button for the video.*]

Text for use with Video Clip D:
Watch the video clip. Then select the tab to order the video clips.
[*Include play button for the video.*]

Text for use on Order the Clips tab:
Order the video clips so that they represent the graph of Marisa's bicycle trip. Explain the ordering of the video clips.

Drag each clip into a box.

[*Present the clips in a row: A, B, C, D. In a row beneath the clips, create four drop boxes, labeled "First", "Second", "Third", and "Fourth". Under this item part, include a response box for the explanation.*]

The draft in Illustration 4.6 requires students to synthesize multiple pieces of information to arrive at a solution. The setting of the task presents the content in a way that could not be done in a traditional item, and the motivating goal of producing a video provides an authentic context. Taken along with potential additional item parts, these features make this task scenario-based.

In the draft item in Illustration 4.6, the focus on Algebra as a content area and the focus on Representing as a NAEP Mathematical Practice were inherent to the scenario-based *Bicycle Trip* task. However, this is not always the case. A scenario-based task may contain items aligned to different content areas and/or NAEP Mathematical Practices, with an identified overarching content area and practice defined by the task problem (i.e., the driving storyline for the task, such as in the *Stacking Chairs* task in Exhibit 4.1).

Bicycle Trip Example: Grade 4 Scenario-Based Task

The NAEP Mathematical Practice of Representing spans all grade levels. Therefore, a set of items inspired by the original NAEP *Bicycle Trip* task can be developed, utilizing the idea of connecting representations at each grade level, 4, 8, and 12. To this end, consider an adaptation of the *Bicycle Trip* item for grade 4. Content that is *not* appropriate to assess at grade 4 is as important to consider as content that is appropriate. For example, although grades 8 and 12 objectives address representations that show change over time, objectives at grade 4 do not.

Since the objective for the grade 8 tasks was Algebra 2.a, Algebra 2.a was initially considered as the objective for the grade 4 task. The guiding question "How can the representation from the original item be adapted to meet the needs of a grade 4 task?" served as a starting point for the completion of the scenario shell shown in Illustration 4.7.

Illustration 4.7. Grade 4 Scenario Shell Adaptation of *Marisa's Bicycle Trip*

Grade	4
Major Content Area(s)	Number Properties and Operations
Context	Ordering video clips of a bicycle trip
Problem	Given a representation and a set of video clips, order the distances indicated in the video clips from least to greatest.
Available Resources and Information	video clips graphical representation
Tools Used	interactive item component
NAEP Mathematical Practice(s)	Representing
NAEP Mathematics Objective(s)	Number Properties and Operations 1.i – Order or compare whole numbers, decimals, or fractions using common denominators or benchmarks.

Note that as the scenario shell developed, the objective was changed from Algebra 2.a to Number Properties and Operations 1.i. This change stemmed from a desire to focus on a provided representation, instead of on translation between representations. To adapt for grade 4, a diagram might be presented showing four locations represented by images. The path Marisa rides connects the images, and each piece of the path is labeled. The video clips can show Marisa riding from one location to the next, indicating the distance between each pair of locations, with each distance measured in the same unit. Students can be asked to order the labels for the pieces of the path by distance, from least to greatest.

Alternatively, if preserving the alignment to Algebra 2.a is essential, the item could require students to match clips with actions, such as "Marisa is speeding up" or "Marisa has stopped." This revision would focus on translating between representational forms.

Bicycle Trip Example: Grade 12 Scenario-Based Task

To reimagine the task for grade 12, content that is not addressed at grade 8 but is addressed at grade 12 was considered first. The comparable grade 12 objective, Algebra 2.a, expands the types of equations used, but does not differentiate the types of interpretations that students are to make. Therefore, a decision was made to increase the complexity of the video clips by including information about speed in each clip, along with a set of clips that cannot be represented by any piece of the graph. The scenario shell for the grade 12 task is shown in Illustration 4.8.

Illustration 4.8. Grade 12 Scenario Shell Adaptation of *Marisa's Bicycle Trip*

Grade	12
Major Content Area(s)	Algebra
Context	Ordering video clips of a bicycle trip
Problem	Given a graph and a set of video clips, order the clips to show Marisa's bicycle trip.
Available Resources and Information	video clips graphical representation
Tools Used	interactive item component
NAEP Mathematical Practice(s)	Representing
NAEP Mathematics Objective(s)	Algebra 2.a – Create and translate between different representations of algebraic expressions, equations, and inequalities (e.g., linear, quadratic, exponential, or *trigonometric) using symbols, graphs, tables, diagrams, or written descriptions.

For the grade 12 task, six video clips can be presented. Two of the clips would show either a speed or an elapsed time that cannot be matched to a piece of the graph. However, each clip would be formatted similarly to provide sufficient context for students to determine speed and/or elapsed time. The item directions would ask students to watch each of the six clips, and then select and order *four* of the clips to show what is most likely Marisa's bicycle trip.

Alternatively, students could view the clips and be asked to make their own graph of Marisa's trip, showing speed versus time. This revision would also focus on translating between representational forms, and, therefore, would also align to Algebra objective 2.a.

Identification and revision of a story concept foundation for a scenario-based task is likely to happen in parallel with the selection of target mathematics objective(s). Concepts serving as candidates for a scenario will likely involve *at least* two actions, such as attending to relationships, visualizing, coordinating, comparing, contrasting, synthesizing, validating, predicting, or persuading via mathematical argument. For example, the original *Bicycle Trip* item involves imagining movement and coordinating between two representations (graphical and verbal). The *Stacking Chairs* adaptation involves attending to relationships, coordinating representations (verbal, symbolic), and predicting (to identify what additional information is needed).

The item type(s) used within a scenario-based task should be based on the structure of the task and the measurement intent. The item types for a scenario-based task will be aligned to the item format that best supports the requested evidence. Therefore, the requirement for developing scoring guides for scenario-based tasks should follow the same principles as outlined in the Item Types section.

What a Scenario-Based Task Is Not

The inclusion of multiple parts is not sufficient to make an item set a scenario-based task. One of the criteria for a task to be scenario-based is that the scenario from which the task is built serves as a driving force through the completion of the task.

The item in Illustration 4.9 contains two parts. A correct response to each part requires use of the table presented at the beginning of the item. While there is a connection between the item parts, there is no underlying storyline driving the mathematical activity required by the item as a whole (also, there are no multimedia aspects and no tools enabled to solve the problem).

Illustration 4.9. Nonexample of a Scenario-Based Task

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
8	Data Analysis, Statistics, and Probability	Representing	Data – 1.c	SCR – composite

VISITORS TO THE MUSEUM AND PARK LAST WEEK

Day	Number of Visitors to the Museum	Number of Visitors to the Park
Monday	63	107
Tuesday	72	93
Wednesday	58	60
Thursday	81	88
Friday	95	64
Saturday	451	208
Sunday	407	225

The table shows the number of visitors who went to the museum and the number of visitors who went to the park on each day last week.

On Tuesday, how many more visitors went to the park than went to the museum?

visitors

On Saturday and Sunday combined, how many more visitors went to the museum than went to the park?

visitors

Scoring Information	
Key	(a) 21 (b) 425

The item in this illustration is based on a NAEP item. The original version of this item appeared in the 2017 grade 4 NAEP Mathematics Assessment with NAEP Item ID 2017-4M9 #3 M3461E1.

While development of scenario-based tasks is a complex and time-consuming process, focusing on the larger aspects of the task prior to development of the items that will comprise the task provides structure within which item writers can work. Additionally, the following considerations can be used to aid the item writer in task development.

- Use of an online environment to create authentic, relevant, and compelling ways of presenting and assessing content and practices.
- Contexts that are interesting to and appropriate for students at the grade level.
- Content and NAEP Mathematical Practices that make sense within the proposed context.
- Content that is out of bounds at a particular grade level, as a check to ensure that the task aligns to on-grade-level objectives.
- Progression of content through grade levels.
- Patience and persistence in iterating the development process and seeking feedback as the task becomes fully formed.

Item development is a complex endeavor involving many components, from designing a content focus given the constraints of a framework, to item format selection, to scoring considerations. To aid in providing structure within which these complexities can be thought through, design patterns have been conceptualized for use as item development tools (Mislevy & Haertel, 2006). Stemming from work in evidence-centered design for assessment, design patterns leverage commonalities in item design so that differing components can be modified (e.g., providing structure for a set of collaborative mathematics tasks that assess different content objectives). As all scenario-based tasks have some common components and some components that vary, consideration should be given to the potential of design patterns to substantially support the development of scenario-based tasks for the 2026 NAEP Mathematics Assessment.

Item Types

Since 1992, the NAEP Mathematics Assessment has used two types of items: multiple choice and constructed response. In 2017, the term “multiple choice” was revised to “selected response” to account for the wider range of item formats available (e.g., matching) with digitally based assessments. Selected response items require a student to select one or more response options from a given, limited set of choices. Constructed response items include those that require students to provide a text-based or numerical response. Both selected response and constructed response items may contain interactive item components (IICs). IICs may be embedded in an item (e.g., virtual ruler) or in the response field (e.g., number line).

Innovative item types made possible by digital test administration are often referred to as technology-enhanced items (TEIs). TEIs have the potential to assess what students know and are able to do in a more authentic way than static selected response items (Sireci & Zenisky, 2006). While item performance indicates that TEIs tend to be more difficult than multiple choice items assessing the same content, both item formats appear to be well correlated with student overall performance on an assessment (Crabtree, 2016). Therefore, TEIs are a middle ground between traditional multiple choice items, which are frequently viewed as artificial but have high reliability, and traditional constructed response items, which allow for more authentic assessment of what students know and can do but are costly in terms of money and time spent during development, administration, and scoring and are likely to have lower reliability (Sireci & Zenisky, 2016).

Research on the development and performance of TEIs is ongoing, but what is known has guided the development of the recommendations in this chapter. As additional item-format-specific research is disseminated, assessment developers will be able to refine development and administration guidelines.

Some selected response items, such as matching or multiple-selection items, have scoring guides to permit partial credit. Every constructed response item has a scoring guide that defines the criteria used to evaluate students' responses. Some short constructed response items can be scored according to guides that permit partial credit, while others are scored as either correct or incorrect. All constructed response scoring guides are refined from work with a sample of actual student responses gathered during item pilot testing. Students are provided information on elements required for a complete response in some of the individual discrete items and in overviews of composite items. This provides all students with greater access to the task and defines the parameters for their responses, honoring their time and energy as they engage in the work.

In 2026, the NAEP Mathematics Assessment retains selected and constructed response item types. The evolving capabilities of digital technology and the addition of NAEP Mathematical Practices mean the 2026 Framework includes the expansion of the two item types to allow for additional object-based and discourse/collaboration-based responses within discrete items and scenario-based tasks.

Selected Response

Selected response items for use on the NAEP Mathematics Assessment include a variety of formats. The listed formats reflect a subset of those with the potential to be developed. Any combination of these item formats in a single item constitutes a composite item.

- Single-selection multiple choice: Students respond by selecting a single choice from a set of given choices.
- Multiple-selection multiple choice: Students respond by selecting two or more choices that meet the condition stated in the stem of the item.
- Matching: Students respond by inserting (i.e., dragging and dropping) one or more source elements (e.g., a graphic) into target fields (e.g., a table).
- Zone: Students respond by selecting one or more regions on a graphic stimulus.
- Grid: Students evaluate mathematical statements or expressions with respect to certain properties. The answer is entered by selecting cells in a table in which rows typically correspond to the statements and columns to the properties checked.
- In-line choice: Students respond by selecting one option from one or more drop-down menus that may appear in various sections of an item.
- Conversational responses (new): Students respond by selecting from two or more choices of conversational responses as part of a discourse-based or collaborative task.

A new selected response item type included for the 2026 NAEP Mathematics Assessment involves the use of discourse and collaboration responses. Items of this type map most directly to the collaborative mathematics and modeling practices outlined in Chapter 3. Current examples ask a student to interact via a text-based scenario with avatars and choose (e.g., through multiple-choice, limited-option selections) from given conversational responses to move the collaborative problem forward. Such a selected response choice then provides some information about the level of collaborative mathematics the student exhibits. Thus, conversational responses retain the structure of selected response item formats and have the potential to be scored polytomously (i.e., some incorrect answer choices may be “more correct” than other incorrect answer choices). Therefore, response options in these items may have differing numbers of score points.

The table in Illustration 4.10 lists and describes selected response item formats, indicates other names by which an item format might be known, and provides the location of exhibits and illustrations within the *Assessment and Item Specifications* of examples and nonexamples. At the beginning of the table are guidelines to assist with the development of selected response items.

Illustration 4.10. Selected Response Item Information

<table>
<tr><td colspan="4">Selected Response (SR) Development Guidelines:
• The item stem includes only the information needed for students to respond.
• Response options are succinctly worded and avoid repetition of phrases in each choice.
• Response options are parallel in mathematical approach and general phrasing.
• Response options do not cue correct responses or use exclusionary language (e.g., always, never).
• Incorrect response options are plausible (e.g., through mathematical conceptions, common errors).
• Incorrect response options connect to the mathematical construct being assessed.
• Rationales are provided for all response options.</td></tr>
<tr><td>NAEP Item Formats</td><td>Similar Item Formats/ Abbreviations</td><td>Student Interaction</td><td>Location(s) of Example Item(s)</td></tr>
<tr><td>single-selection multiple choice</td><td>multiple choice (MC)</td><td>Student selects one of four given response options at grade 4. At grades 8 and 12, student selects one of five response options.</td><td>Illustration 3.1
Illustration 3.3</td></tr>
<tr><td>multiple-selection multiple choice</td><td>multiple select (MS)</td><td>Student selects two or more of the given response options.</td><td>Illustration 4.11</td></tr>
<tr><td>matching</td><td>drag and drop
gap match</td><td>Student inserts one or more source elements (e.g., graphics) into target fields (e.g., cells of a table).</td><td>Illustration 3.22
Illustration 4.12</td></tr>
<tr><td>zone</td><td>hot spot (HS)</td><td>Students respond by selecting one or more regions on a graphic stimulus.</td><td>Illustration 4.13a
Illustration 4.13b</td></tr>
<tr><td>grid</td><td>matching table</td><td>Students evaluate mathematical statements or expressions with respect to certain criteria. The response is entered by selecting cells in a table in which rows typically correspond to the statements and columns to the properties checked.</td><td>Illustration 3.8
Illustration 4.14</td></tr>
<tr><td>in-line choice</td><td>in-line dropdown (IC)</td><td>Students respond by selecting one option from one or more drop-down menus that appear in various sections of an item.</td><td>Illustration 4.15</td></tr>
</table>

Single-Selection Multiple Choice. Multiple choice items are an efficient way to assess knowledge and skills, and they can be developed to measure various levels of rigor. In a well-designed multiple choice item, the stem clearly presents the problem to the student. The stem may be in the form of a question, a phrase, or a mathematical expression, as long as it conveys what is expected of the student. Historically, in NAEP, the stem is followed by either four or five response options, only one of which is correct. The item in Illustration 3.1 in Chapter 3 illustrates a straightforward stem with a direct question. The distractors are plausible, but only one response option is correct.

Multiple-Selection Multiple Choice. As with single-selection multiple choice items, the stem of a well-designed multiple-selection multiple choice item clearly presents the problem to the student. The stem may be in the form of a question, a phrase, or a mathematical expression, as long as it conveys what is expected of the student. To avoid confusion for students, it is common in assessment development that the stem in multiple-selection items is followed by *more than* four response options with *more than* one correct response option (e.g., when single-selection items on the same assessment have four options with exactly one option correct). Directions for this item format should indicate either the number of correct responses or that students should select all of the correct responses. Due to the selection of multiple responses, some items allow for partial credit. For these items, scoring guides are developed to indicate how the partial credit is allocated.

Correctly responding to items using the multiple-selection format is more challenging than single-selection multiple choice items, as students must determine not only the relationship between a response and the item stem, but also the relationships among the response options (Baghaei & Dourakhshan, 2016). The item in Illustration 4.11 asks students to select all of the response options that represent a unit of measure for the length of time a person will drive. Using a multiple-selection multiple choice item format allows for the assessment of student recognition of more than one appropriate unit, changing the measurement intent from that of an item asking students to identify and select exactly one unit of measure.

Illustration 4.11. Selected Response Example: **Multiple-Selection Multiple Choice Item**

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
4	Measurement	Other	Meas – 2.a	SR – MS

Ms. Taylor will drive from Maine to Florida.

Which of the following units of measurement could be used to measure the amount of time it will take her to complete the drive?

Select all the correct answers.

- A ☐ Days
- B ☐ Gallons
- C ☐ Hours
- D ☐ Miles
- E ☐ Pounds
- F ☐ Yards

Clear Answer

Scoring Information	
Key	A, C
Correct	Two correct selections and no incorrect selections
Partial	One correct selection and no incorrect selections
Incorrect	Two or fewer correct selections and one or more incorrect selections

The item in this illustration is based on a NAEP item. The original version of this item appeared in the 2017 grade 4 NAEP Mathematics Assessment with NAEP Item ID 2017-4M1 #7 M3706MS.

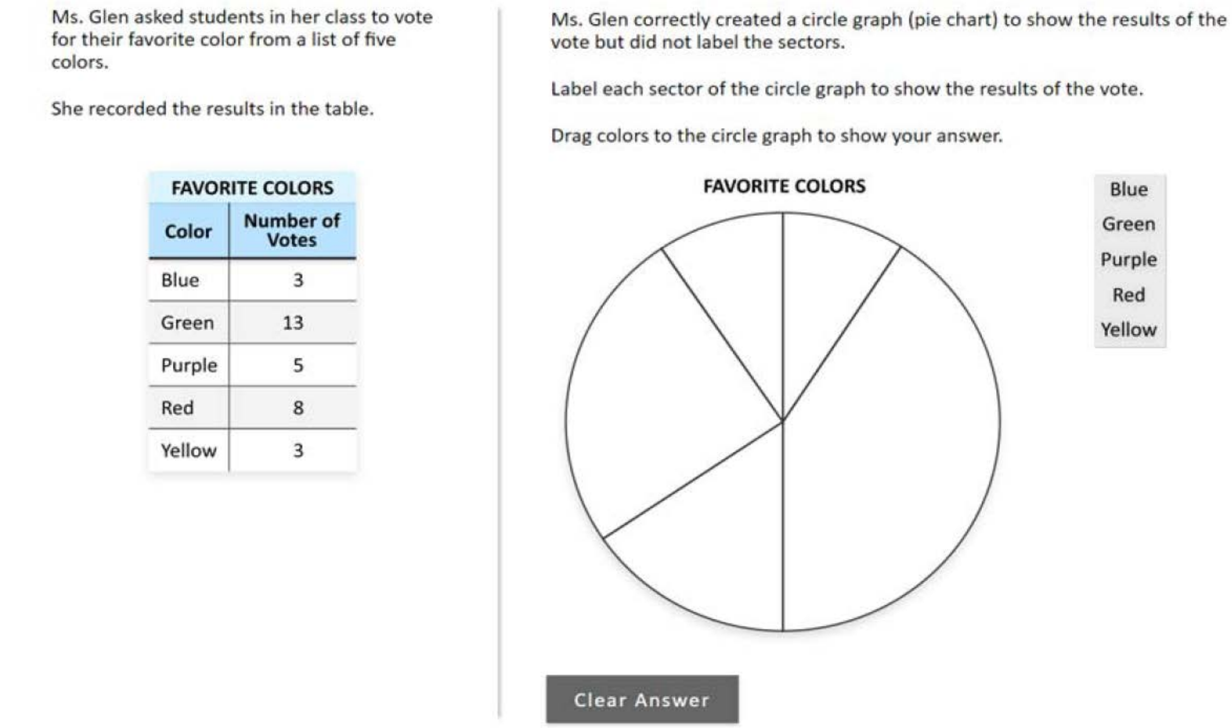
Matching. Matching items take many forms, but each involves the dragging and dropping of one or more objects. For example, a matching item may require the dragging of text, numbers, or figures into indicated spaces; the ordering of presented text, numbers, or figures; or the matching of a subset of objects from one set of information to objects in another set.

Matching items can quickly become quite complicated, based on the number of dragging and dropping actions required. In addition to accessibility concerns, item writers should consider the number of actions in light of the measurement intent of the item – that is, how much information students need to provide to demonstrate evidence of understanding of the assessed objective. Additionally, when possible, the development of more objects to drag than locations in which to drop them tends to allow students to make an error in one placement without impacting the other placements (see the Accessibility section for more on the related topic of universal design).

The item in Illustration 4.12 asks students to drag each color into the correct piece of the circle graph. As each color is required to be represented in the circle graph, a one-to-one relationship between the colors and the pieces of the graph is the necessary structure.

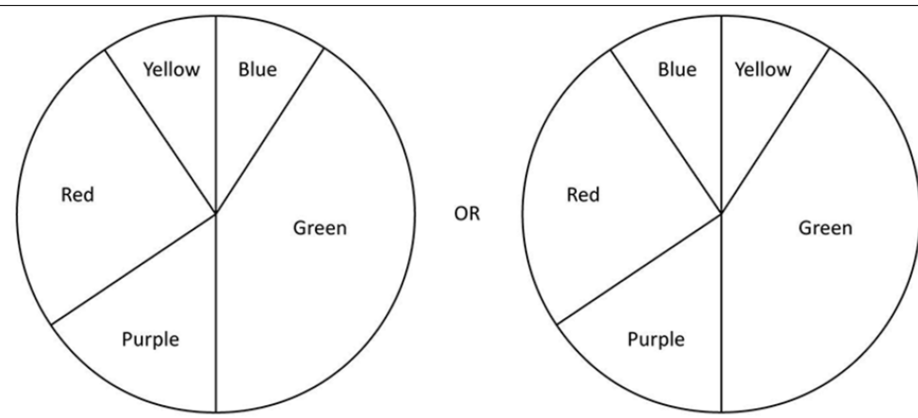
Illustration 4.12. Selected Response Example: Matching Item

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
8	Data Analysis, Statistics, and Probability	Representing	Data – 1.b	SR – matching



Scoring Information

Key



Correct	Five colors correctly placed
Incorrect	Incorrect response

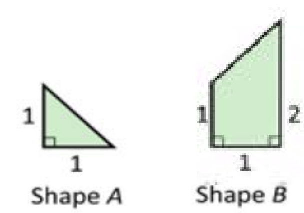
The item in this illustration is based on a NAEP item. The original version of this item appeared in the 2017 grade 8 NAEP Mathematics Assessment with NAEP Item ID 2017-8M3 #2 M3806MS.

Zone. Zone items involve the selection of a graphic or graphics or the selection of a location or locations on a graphic. The zone item format can take the place of some drawing activities encountered on some paper-and-pencil assessments, such as plotting a point on a number line. As with matching items, writers should consider the number and type of student actions required in light of accessibility and the measurement intent of the item. When developing an item that requires the selection of graphics, consideration should be given to the number of graphics presented and the number of correct graphics. When developing an item that requires the selection of a location or locations on a graphic, consideration should be given to the size and clarity of the graphic, the number of locations that are selectable, and the number of correct locations. For zone items, the selectable locations should be purposeful and clearly defined.

The item in Illustration 4.13a presents a set of six graphics from which students select. Since two of the six graphics are correct, this item is comparable to a multiple-selection multiple choice item. Note that side lengths and right-angle markings are used to clearly convey the size and shape of each figure.

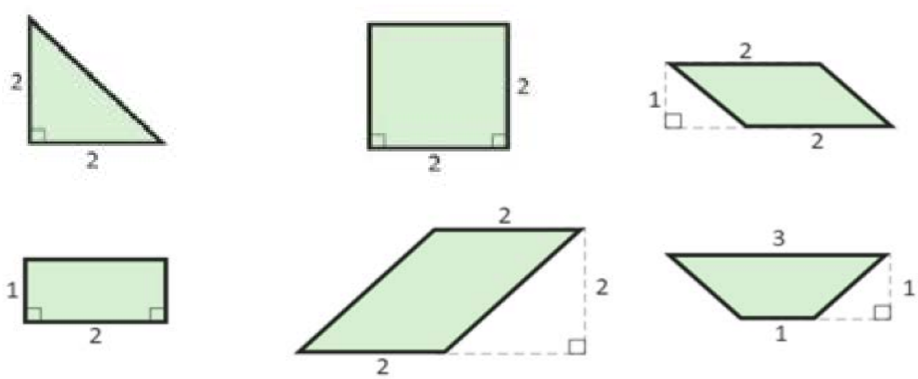
Illustration 4.13a. Selected Response Example: **Grade 8 Zone Item**

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
8	Geometry	Other	Geom – 2.d	SR – zone



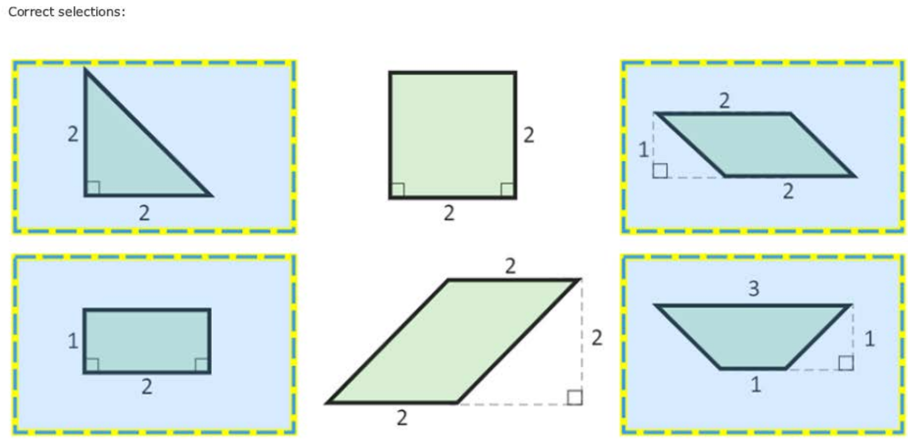
Which of the following figures can be formed by placing shape *A* adjacent to shape *B* so that they share a side?

Select all the correct answers.



Clear Answer

Illustration 4.13a. **Selected Response Example:** **Grade 8 Zone Item (continued)**

Scoring Information	
Key	
Correct	Four correct selections and no incorrect selections
Partial	Four correct selections and one incorrect selection OR Three correct selections and no incorrect selections
Incorrect	Four correct selections and more than one incorrect selection OR Fewer than four correct selections and one or more incorrect selections OR Fewer than three correct selections

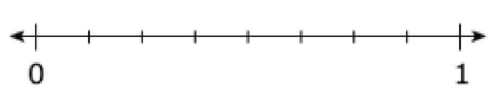
The item in this illustration is based on a NAEP item. The original version of this item appeared in the 2017 grade 8 NAEP Mathematics Assessment with NAEP Item ID 2017-8M3 #12 M3814EM.

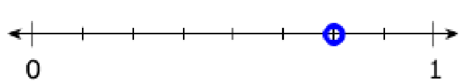
The item in Illustration 4.13b presents a number line on which students can select a point. Although information regarding the numbers and locations of the zones is not provided, it is likely that each of the hash marks on the number line is a zone. This placement of the zones allows students to select any eighth without concern over selection of a zone between two hash marks, approximating an equivalent fraction with a denominator other than 8, or concerns over student dexterity when selecting a zone.

Illustration 4.13b. **Selected Response Example:** **Grade 4 Zone Item**

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
4	Number Properties & Operations	Representing	Num – 1.h	SR – zone

Select a point on the number line to plot a point that is equivalent to $\frac{3}{4}$.



Scoring Information	
Key	

The item in this illustration is based on a 2017 PARCC item with Item ID VF889661, aligned to evidence statement 3.NF.3a-2.

Grid. Grid items involve the selection of cells in a table to indicate a response. The rows of the table contain stimuli to be considered. The stimuli should be mathematically related. The first cell in each column of the table lists the options from which students choose. The options should be plausible for each stimulus. As with previously discussed item formats, writers should consider the number and type of student actions required in light of accessibility and the measurement intent of the item – that is, how much information students need to provide to demonstrate evidence of understanding of the assessed objective. This should inform the number of rows and columns included in an item.

The item in Illustration 4.14 presents a set of four measurements as stimuli and two comparisons as choices. With the comparison of measurements assessed by this item, similar thinking can be applied for each stimulus. However, the nature of the stimuli chosen requires consideration for each case, as each stimulus is independent of the others.

Illustration 4.14. Selected Response Example: Grid Item

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
8	Measurement	Other	Meas – 2.b	SR – grid

2 pints = 1 quart
4 quarts = 1 gallon

There are 2 pints in a quart and there are 4 quarts in a gallon.

Which of the following measures are less than 5 quarts and which are more than 5 quarts?

Make one selection for each measure to show your answer.

Measures	Less than 5 Quarts	More than 5 Quarts
5 pints	○	○
5 gallons	○	○
1 gallon and 1 pint	○	○
1 gallon and 5 pints	○	○

Clear Answer

Scoring Information

Key

Correct selections:

Measures	Less than 5 Quarts	More than 5 Quarts
5 pints	X	
5 gallons		X
1 gallon and 1 pint	X	
1 gallon and 5 pints		X

Correct	Four correct selections
Partial	Three correct selections
Incorrect	Fewer than three correct selections

The item in this illustration is based on a NAEP item. The original version of this item appeared in the 2017 grade 8 NAEP Mathematics Assessment with Item ID 2017-8M3 #5 M3838MS.

In-Line Choice. In-line choice items require students to select text that correctly completes a statement. Typically, the item stem presents information relevant to the completion of one or more statements. The statements are written beneath the stem, with drop-down menus that present plausible options for sentence completion. Item writers should take care when determining the number of options for each drop-down menu, as the total number of response options has the potential to impact the amount of reasoning required for students to complete the item. Additionally, in terms of accessibility, a student taking the test with a screen reader must listen to *every* potential answer, so the number of options in each drop-down menu impacts the number of combinations that the student must hear and manage.

The item in Illustration 4.15 provides information about two functions. Following the information, two statements containing drop-down menus are given. The first statement asks students to compare the slopes of the two functions. The second statement asks students to compare the *y*-intercepts of the two functions. In this example, the option that completes one statement is independent of the option that completes the other statement.

Illustration 4.15. Selected Response Example: In-Line Choice Item

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
8	Algebra	Representing	Alg – 2.b	SR – IC

Function A is a linear function that passes through the points shown in the table.

Function A

x	y
6	7
−3	4

Function B is a linear function defined by the equation $y = 2x + 1$.

Select from the drop-down menus to correctly complete each sentence.

The slope of Function A is [Choose... ▾] the slope of Function B.

- greater than
- less than
- equal to

The *y*-intercept of Function A is [Choose... ▾] the *y*-intercept of Function B.

- greater than
- less than
- equal to

Scoring Information

Key	The slope of Function A is [less than ▾] the slope of Function B. The *y*-intercept of Function A is [greater than ▾] the *y*-intercept of Function B.

The item in this illustration is based on a 2019 PARCC item with Item ID VH139356, aligned to evidence statement 8.F.2.

Constructed Response

Constructed response items for the NAEP Mathematics Assessment also include a variety of formats, including those listed below. Any combination of constructed response item formats or selected response formats with at least one constructed response format in a single item constitutes a composite constructed response item.

- Short constructed response: Students respond by giving either a numerical result or the correct name or classification for a group of mathematical objects, or possibly by writing a brief explanation for a given result.
- Extended constructed response: Students respond by giving a description of a situation, an analysis of a graph or table of values or an algebraic expression, or a computation involving specific numerical values. These items require students to consider a situation that requires more than a numerical response or a short verbal communication.
- Object-based responses (new): Students respond by manipulating or using a physical object. The state of the object upon item completion is the response (more on this type of response is presented in the discussion of Illustration 4.19).

The table in Illustration 4.16 describes constructed response item formats, indicates other names by which an item format might be known, and provides the locations of exhibits and illustrations within this document of examples and nonexamples. At the beginning of the table are guidelines to assist with the development of constructed response items.

Illustration 4.16. Constructed Response Item Information

<table>
<tr><td colspan="4">Constructed Response (CR)
Best used when student communication of the correct response and/or support for a response provides greater evidence than use of other item types.
Examples of item structures or response requirements for which CR items are appropriate are
• computational fluency,
• writing an equation to model a situation, and
• justifying a mathematical claim.</td></tr>
<tr><th>NAEP Item Formats</th><th>Abbreviations</th><th>Description</th><th>Location(s) of Example Item(s)</th></tr>
<tr><td>short constructed response</td><td>SCR</td><td>Ask students to give either a numerical result or the correct name or classification for a group of mathematical objects, draw an example of a given concept, or possibly write a brief explanation for a given result.</td><td>Exhibit 3.7
Illustration 4.17a
Illustration 4.17b
Illustration 4.17c</td></tr>
<tr><td>extended constructed response</td><td>ECR</td><td>Ask students to solve a problem by applying and integrating mathematical concepts and require students to analyze a mathematical situation and explain a concept, or both.</td><td>Illustration 4.18a
Illustration 4.18b</td></tr>
<tr><td>object-based response</td><td></td><td>Ask students to manipulate or use a physical object to provide a response.</td><td>Illustration 4.19</td></tr>
</table>

Every constructed response item has a scoring guide that defines the criteria used to evaluate students' responses. Some short constructed response items can be scored according to guides that permit partial credit, while others are scored as either correct or incorrect. All constructed response scoring guides are refined from work with a sample of actual student responses gathered during pilot testing of items. Students are provided information on elements required for a complete response in individual discrete item stems and/or in overviews of composite items. This provides all students with greater access to the item and defines the parameters for their response, honoring their time and energy as they engage in the work.

The type of constructed response item, short or extended, should depend on the mathematical construct being assessed – the content of the objective, the NAEP Mathematical Practice(s) addressed, and the rigor involved in determining and constructing a solution. Item writers should draft a scoring rubric as they are developing the item, so that both the item and the rubric reflect the construct being measured.

In developing the scoring rubric for an item, writers should think about what kind of student responses would show increasing degrees of knowledge and understanding (e.g., as outlined in the ALDs). Writers should sketch condensed sample responses for each score category, even before pilot use. Similarly, a mathematical justification or explanation for each category in a rubric description is needed. Doing so scaffolds development of a clear scoring rubric and provides guidance for those scoring the item. Item writers should refer to additional directions for developing scoring guides, provided by Governing Board policy and the assessment development contractor, when constructing scoring information for an item.

Short Constructed Response. To provide more reliable and valid opportunities for extrapolating about students' approaches to problems, NAEP assessments include items referred to as short constructed response (SCR) items. These are short-answer items that require students to give a numerical result or the correct name or classification for a group of mathematical objects, draw an example of a given concept, or possibly write a brief explanation for a given result. SCR items may be scored as correct, incorrect, or partially correct, depending on the nature of the problem and the information gained from students' responses.

Most fill-in-the-blank (FIB) items with one response box are SCR items. FIB items require students to enter a numerical or short verbal text (e.g., a name). Some FIBs are written to be scored dichotomously; that is, with two scoring categories: correct or incorrect. FIBs with two scoring categories should measure knowledge and skills in a way that multiple choice items cannot, or be designed to elicit greater evidence of students' understanding. Such FIBs might be appropriate for measuring computation skills, for example, to avoid guessing or estimation (which could be a factor if a multiple choice item were used). FIB items are also useful when there is more than one possible correct answer or when there are different ways to display an answer. Item writers should take care that FIB items would not be better or more efficiently structured as multiple choice items; there should be a purpose for the use of the item type, based on the measurement intent of the item.

Item writers should draft a scoring rubric for each FIB. A writer will not necessarily need to determine the scoring categories for an item, as this depends on the robustness of the item as determined in an iterative item development process.

For dichotomous items, the rubrics should define the following two categories: Correct and Incorrect. The item in Illustration 4.17a requires students to perform a calculation. Since this item assesses computational skills, the use of the FIB format is appropriate. The scoring information provided defines a correct result, indicating what is required for a correct response and for an incorrect response.

Illustration 4.17a. Short Constructed Response Example: Fill-in-the-Blank Item

<table>
<tr><th>Grade Level</th><th>Content Area</th><th>Assessed Practice(s)</th><th>Objective ID</th><th>Item Format</th></tr>
<tr><td>4</td><td>Number Properties and Operations</td><td>Other</td><td>Num – 3.c</td><td>SCR – FIB</td></tr>
<tr><td colspan="5">Divide.

$228 \div 4 = \boxed{}$</td></tr>
<tr><th colspan="5">Scoring Information</th></tr>
<tr><th>Key</th><td colspan="4">57</td></tr>
<tr><th>Correct</th><td colspan="4">Answer of 57</td></tr>
<tr><th>Incorrect</th><td colspan="4">Incorrect response</td></tr>
</table>

The item in this illustration is based on a NAEP item. The original version of this item appeared in the 2017 grade 4 NAEP Mathematics Assessment with NAEP Item ID 2017-4M1 #4 M367801.

Some FIBs are written to be scored on a three-category scale. These items should measure knowledge and skills that require students to go beyond giving a viable answer, allowing for degrees of accuracy in a response so that a student can receive some credit for demonstrating partial understanding of the concept or skill measured by the item.

For items with three score categories, the rubrics should define the following categories: Correct, Partial, and Incorrect. The item in Illustration 4.17b is an FIB item that asks students to complete the cells of a table. The use of the FIB format allows this item to occupy less space than it would have if students had been required to select one of four tables presented as response options. This item was developed with three score categories. A correct response requires that all of the cells be completed correctly, and a partial score is presented for an answer that demonstrates some understanding of how to extend the relationship given.

Illustration 4.17b. Short Constructed Response Example: Fill in Multiple Cells in a Table

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
8	Algebra	Other	Alg – 1.a	SCR – FIB

Melissa saves money for six weeks to buy a sweater.

She records her weekly savings.

She saves $2.50 the first week.

Each week, she saves $1.25 more than she saved the previous week.

Complete the table to show how much Melissa saves each week.

MELISSA'S SAVINGS BY WEEK

Week	Money Saved ($)
1	2.50
2	
3	
4	
5	
6	

Illustration 4.17b. Short Constructed Response Example: Fill in (continued)

<table>
<tr><th colspan="2">Scoring Information</th></tr>
<tr><td>Key</td><td>
<table>
<tr><th colspan="2">MELISSA'S SAVINGS BY WEEK</th></tr>
<tr><th>Week</th><th>Money Saved ($)</th></tr>
<tr><td>1</td><td>2.50</td></tr>
<tr><td>2</td><td>3.75</td></tr>
<tr><td>3</td><td>5.00</td></tr>
<tr><td>4</td><td>6.25</td></tr>
<tr><td>5</td><td>7.50</td></tr>
<tr><td>6</td><td>8.75</td></tr>
</table>
Note: Accept equivalent values</td></tr>
<tr><td>Correct</td><td>Correct response</td></tr>
<tr><td>Partial</td><td>4 of 5 terms correct
OR
Rule applied correctly to all but one term
OR
Response shows a correct cumulative total for each week</td></tr>
<tr><td>Incorrect</td><td>Incorrect response</td></tr>
</table>

The item in this illustration is based on a NAEP item. The original version of this item appeared in the 2017 grade 8 NAEP Mathematics Assessment with NAEP Item ID 2017-8M9 #7 M3553E1.

Some SCR items require students to enter more than one or two words into a provided answer block (e.g., a brief explanation for a given result). The item in Illustration 4.17c was previously introduced in Chapter 2 (Illustration 2.3). This item is presented again here with scoring information. Note that, like the item in Illustration 4.17b, this item was developed with three score categories.

Illustration 4.17c. Short Constructed Response Example

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
8	Data Analysis, Statistics, and Probability	Other	Data – 4.e	SCR – composite

Al, Bev, and Carmen are going on a ride at the park. Only 2 people can go on the ride at a time. They can pair up 3 different ways, as shown below.

Al and Bev

Al and Carmen

Bev and Carmen

Derek decides to join the group. How many different ways can the 4 students pair up?

Answer: ___________________

Show your work or explain how you got your answer.

Scoring Information

Key	6 ways: Al and Bev Al and Carmen Al and Derek Bev and Carmen Bev and Derek Carmen and Derek The supporting work or explanation should show or explain how the pairings of people were obtained; this may include drawings only, words only, or a combination of both.
Correct	Correct response 6 different ways with justification that demonstrates how the four people would be paired. It is possible to justify the answer of 6 without explicitly stating the 6 pairs by name, but the justification needs to be clear.
Partial	Partially correct response Response contains the 6 different ways, but the justification either is missing or is partially correct or partially complete. The partial justification may demonstrate that Derek can be paired with more than just one of the remaining people, but the justification falls short of complete, as long as the work shown does not demonstrate that 6 was obtained via invalid reasoning, should also be placed here. OR Response does NOT obtain 6 ways but does demonstrate in some way that Derek can be paired with more than just one of the remaining people.
Incorrect	Incorrect response

The item in this illustration is based on a NAEP item. The original version of this item appeared in the 2009 grade 4 NAEP Mathematics Assessment with NAEP Item ID 2013-4M6 #14 M136901.

Extended Constructed Response. Extended constructed response items require a greater amount of mathematical rigor than short constructed response items. In general, extended constructed response items ask students to solve a problem by applying and integrating mathematical concepts, require students to analyze a mathematical situation and explain a concept, or both. These items should be developed so that the knowledge and skills they measure are worth the additional time and effort that they take the student to respond and the time and effort that scoring the response takes. Extended constructed response items typically have five scoring categories: Extended, Satisfactory, Partial, Minimal, and Incorrect. In some cases, it may be appropriate to have four scoring categories for an extended constructed response item, depending upon the construct assessed and the nature of expected student responses to the item.

The items in Illustrations 4.18a and 4.18b are extended constructed response items. The item in Illustration 4.18a asks students to read and interpret two graphical representations of the same data. The item consists of two parts: a multiple-selection multiple choice item part and an FIB item part. The scoring rubric for this item consists of five scoring categories. For Extended credit, a complete and correct response must be provided for both item parts, while Satisfactory credit allows for a minor error. Responses scored as Partial, Minimal, and Incorrect show decreasing levels of correctness.

Illustration 4.18a. Extended Constructed Response Example: MS and FIB Item Parts

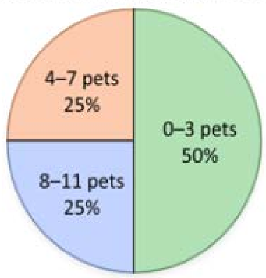
Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
8	Data Analysis, Statistics, and Probability	Representing	Data – 1.c	ECR – composite

A teacher surveyed 8 students in a class to find out how many pets they each have.

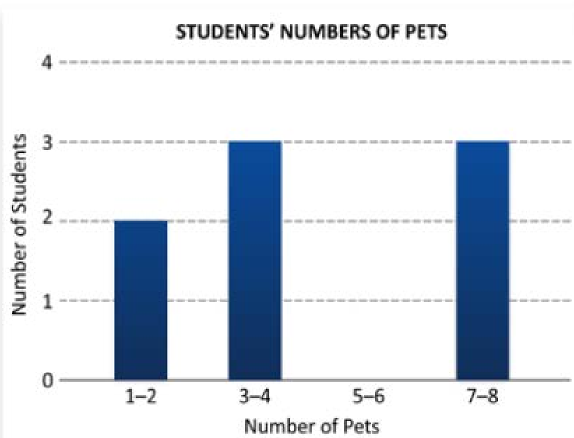
Boyd and Jenny represented the results of the survey in two different ways.

Boyd made a circle graph (pie chart) to show the results.

STUDENTS' NUMBERS OF PETS



Jenny made a bar graph to show the results.



Based on the circle graph, which of the following sets of data could be the results of the survey?

Select all the correct answers.

- A ☐ 0, 1, 2, 2, 3, 3, 5, 10
- B ☐ 0, 1, 2, 3, 5, 6, 9, 11
- C ☐ 0, 2, 2, 3, 4, 8, 9, 10
- D ☐ 1, 1, 1, 1, 4, 4, 8, 8
- E ☐ 1, 2, 4, 5, 6, 7, 8, 10

Clear Answer

Based on both Boyd's circle graph and Jenny's bar graph, what could be the results of the survey?

Enter a number in each box to show how many pets each student could have.

Illustration 4.18a. Extended Constructed Response: MS and FIB Item Parts (continued)

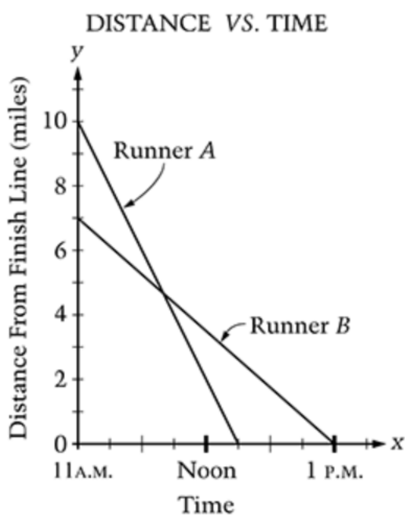
Scoring Information	
Key	(a) Correct selections: B. 0, 1, 2, 3, 5, 6, 9, 11 D. 1, 1, 1, 1, 4, 4, 8, 8 (b) Answer: 1, 1, 3, 3, 4, 7, 8, 8 OR 1, 2, 3, 3, 4, 7, 8, 8 OR 2, 2, 3, 3, 4, 7, 8, 8
Extended	Two correct selections and no incorrect selections for part (a) with a correct data set for part (b)
Satisfactory	Two correct selections and one incorrect selection for part (a) with a correct data set for part (b) OR One correct selection and no incorrect selections for part (a) with a correct data set for part (b)
Partial	Two correct selections and no incorrect selections for part (a) with an incorrect data set for part (b) OR Two correct selections and more than one incorrect selection for part (a) with a correct data set for part (b) OR One correct selection and one or more incorrect selections for part (a) with a correct data set for part (b) OR No correct selections for part (a) with a correct data set for part (b)
Minimal	Two correct selections and one incorrect selection for part (a) with an incorrect data set for part (b) OR One correct selection and no incorrect selections for part (a) with an incorrect data set for part (b)
Incorrect	Two correct selections and more than one incorrect selection for part (a) with an incorrect data set for part (b) OR One correct selection and one or more incorrect selections for part (a) with an incorrect data set for part (b) OR No correct selections for part (a) with an incorrect data set for part (b)

The item in this illustration is based on a NAEP item. The original version of this item appeared in the 2017 grade 8 NAEP Mathematics Assessment with NAEP Item ID 2017-8M3 #13 M3859CL.

The item in Illustration 4.18b asks students to interpret three characteristics of a graph. Unlike the item in Illustration 4.18a, scoring for this item is by characteristic. That is, there are three score categories for each characteristic. Since the item requires words and numbers for a complete response, partial credit addresses scoring for a response that includes only words or only numbers.

Illustration 4.18b. Extended Constructed Response Example: Extended Text, Multi-Response

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
12	Algebra	Representing	Alg – 4.d	ECR – Composite



The graph above shows distance *versus* time for a race between runners A and B. The race is already in progress, and the graph shows only the portion of the race that occurred after 11 A.M.

The table shown lists several characteristics of the graph. Interpret these characteristics in terms of what happened during this portion of the race. Include times and distances to support your interpretation. (A sample interpretation of the y-intercepts is given in the table.)

Characteristic of Graph	Interpretation in Terms of the Race
y-intercepts	At 11 A.M. Runner A is 10 miles from the finish line and Runner B is 7 miles from the finish line.
Slopes	
Point of intersection	
x-intercepts	

Illustration 4.18b. Extended Constructed Response Example (continued)

<table>
<tr><th colspan="2">Scoring Information</th></tr>
<tr><td>Key</td><td>
There are three components in this response.
<table>
<tr><th>Characteristic of Graph</th><th>Interpretation in Terms of the Race</th></tr>
<tr><td>y-intercepts</td><td>At 11 A.M. Runner A is 10 miles from the finish line and Runner B is 7 miles from the finish line.</td></tr>
<tr><td>(a) Slopes</td><td>Runner A's speed is 8 mph and Runner B's speed is 3.5 mph.</td></tr>
<tr><td>(b) Point of intersection</td><td>Runner A and Runner B are both $4\frac{2}{3}$ miles from the finish line at 11:40 A.M.
NOTES:
• Accept distances from 4.5 to 5 miles, inclusive and times from 11:30 to 11:45, inclusive.
• This can also be interpreted as the time and distance when Runner A overtakes Runner B.</td></tr>
<tr><td>(c) x-intercepts</td><td>Runner A finishes the race at 12:15 P.M. and Runner B finishes the race at 1 P.M.</td></tr>
</table>
</td></tr>
<tr><td>Part A Correct</td><td>Acceptable response</td></tr>
<tr><td>Part A Partial</td><td>
Acceptable interpretation without numerical values for slopes
OR
Numerical values without acceptable interpretation for slopes
Examples:
<table>
<tr><th>Characteristic of Graph</th><th>Interpretation Without Values</th><th>Values Only for Interpretation</th></tr>
<tr><td>Slopes</td><td>The speed of each runner.</td><td>A = 8 mph
B = 3.5 mph</td></tr>
</table>
</td></tr>
<tr><td>Part B Correct</td><td>Acceptable response</td></tr>
<tr><td>Part B Partial</td><td>
Acceptable interpretation without numerical values for point of intersection
OR
Numerical values without acceptable interpretation for point of intersection
Examples:
<table>
<tr><th>Characteristic of Graph</th><th>Interpretation Without Values</th><th>Values Only for Interpretation</th></tr>
<tr><td>Point of intersection</td><td>The point where or when Runner A overtakes Runner B.
OR
The time when both runners are at the same distance from the finish line.</td><td>11:40AM
$4\frac{2}{3}$ miles</td></tr>
</table>
</td></tr>
<tr><td>Part C Correct</td><td>Acceptable response</td></tr>
<tr><td>Part C Partial</td><td>
Acceptable interpretation without numerical values for x-intercepts
OR
Numerical values without acceptable interpretation for x-intercepts
Examples:
<table>
<tr><th>Characteristic of Graph</th><th>Interpretation Without Values</th><th>Values Only for Interpretation</th></tr>
<tr><td>x-intercepts</td><td>The end of the race.
OR
The time that each runner finishes the race.</td><td>A = 12:15 PM
B = 1:00 PM</td></tr>
</table>
</td></tr>
</table>

The item in this illustration is based on a NAEP item. The original versions of this item appeared in the 2009 grade 12 NAEP Mathematics Assessment with NAEP Item IDs 2009-12M2 #9 M1809CL, M180901, M180902, and M180903.

Object-Based Response. The digitally based NAEP Mathematics Assessment already incorporates use of virtual tools in tool-based responses (e.g., on-screen rulers). A new item type for NAEP Mathematics Assessments in 2026 and beyond is object-based responses. There is a growing ability to capture how students use manipulatives, both digital on-screen and with "smart" physical objects off-screen that can monitor activity and be connected to the digital assessment. Here there are at least two opportunities to be forward-thinking. First, further inquiry is warranted into ways to incorporate physical manipulatives that can collect data mapped to assessed constructs. The advances in smart tool technology are particularly suited to directly capture the NAEP Mathematical Practices outlined in Chapter 3. Second, further work is needed to align the data collected from tasks to valid measures of a construct. For example, one could imagine students manipulating a physical object, and the solution states that they come up with at different points in time (since activity is monitored continuously) could provide strong differentiating information about mathematical modeling. A solution state of the physical orientation of an object would be the answer (versus a discrete selection or clicking a multiple choice option). These and other opportunities will help NAEP move toward the ultimate goal of using tasks in the assessment in ways that capture the variety of ways students know and do mathematics.

As noted, the state of the object defines an object-based response. To collect evidence about the content being assessed by an item involving an object, the response provided by the state of the object must indicate enactment of the mathematics in the content objective. For example, consider an item that aims to assess angle measurement, for which students have a physical protractor. A response indicated by the protractor aligned correctly to measure an angle would not provide sufficient evidence that the student can read the protractor to determine the angle measurement. Therefore, this would not be an object-based response item (though, if the protractor were virtual, the item could be a digital tool–based response item, such as some items currently used on the NAEP Mathematics Assessment). In contrast, an item that asks students to represent the number 126 with base 10 blocks, where students manipulate physical "smart" base 10 blocks, would collect evidence that the student can represent a number in base 10. The submitted state of the base 10 blocks would be an object-based response. Potential objects for use on future NAEP Mathematics Assessments, should they be developed as smart objects, are blocks or tiles for representing bases other than base 10, fraction strips or bars, integer chips, and algebra tiles. Additional smart objects might also be considered as the technology of the assessment evolves.

The item in Illustration 4.19 is based on a NAEP item that was accompanied by physical shapes which students manipulated to create their response. To respond to the item, students were asked to trace the shape that resulted from physically manipulating the provided pieces. The provided shapes in an object-based response item could be "smart" physical objects the students would manipulate and whose final state would be captured as the response to the item.

Illustration 4.19. Foundation for an Object-Based Response Item

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
4	Geometry	Representing	Geom – 3.b	Object-Based Response
[Students are provided a set of physical shapes. Among them are congruent right triangles, each labeled *Q*.] You will need two shapes labeled *Q*. Please find those two shapes now. Use the two shapes labeled *Q* to make a square. [The functionality that allows the shape the student constructs to be captured will determine the completion of the item.]				

The item in this illustration is based on a NAEP item. The original version of this item appeared in the 1996 grade 4 NAEP Mathematics Assessment with NAEP Item ID 1996-4M10 #3 M061903.

Potential Scoring Advances

With the rapid advances in natural language processing, in the future there may be potential for mathematical collaboration to be assessed more effectively in open-ended constructed response formats. For example, the assessment might ask for and then automatically code responses where students are asked to explain their thinking or justify a contribution to collaborative mathematics. While not available at the time of the 2026 Framework revision, such technology may become available for future administrations of the NAEP Mathematics Assessment and may increase accessibility. The assessment might ask students to input their thinking or dialogue via voice (with automatic transcription into text for coding and analysis), which would dramatically open up ways for students to demonstrate what they know and can do. Similarly, pairs of students might be asked to turn on an audio documentation (e.g., a recording device) as they work together on a modeling task. The record of discourse would be part of assessment response, measurable evidence of students creating representations, making conjectures, critiquing and debating, revoicing, or justifying their solutions to one another. Considerable research and development work are needed around the technology for natural language processing and related domains, combined with careful mapping to constructs and measurement needs, to realize the aspirational goal of opening up such ways for students to show what they can do mathematically. Also, special attention must be paid to issues of consent and privacy when considering voice recording.

Additional Scoring Guide Development Information

NAEP scoring guides will be developed in accordance with recommended practice and the Governing Board Item Development and Review Policy (2002). The Board's policy includes principles about scoring guides that apply to all NAEP assessments.

Composite Items

Composite items are composed of two or more item parts. Any item format can be used in a composite item. Some examples of composite items from this chapter are located in Illustration 4.9, which utilizes the FIB item format in each of the two parts, and Illustration 4.18a, which utilizes multiple-selection multiple choice and FIB item formats.

Response Data and Process Data for Future NAEP Mathematics Assessments

A key challenge is the need to capture enough information about mathematics content and practices for a reliable and valid assessment. When this happens, within the context of scenario-based tasks, which require more time for engagement and completion, data may be available from fewer items per student.

An opportunity for future NAEP Mathematics Assessments is to develop validated measures from process data, which is generated based on student interaction with the tools and systems in the scenario-based tasks (e.g., clickstreams or activity logs). The data are different from what might be generated in a non-digital format, so it is necessary to describe how the additional data might be handled.

Conventional items always involve the student in a direct response, which generates *response data*. For example, after being presented with information in a table, the student is asked a text-based question and given a limited set of choices from which to select an answer. Student direct responses can also be used in scenarios. Direct response data can include selection from a set of choices (e.g., multiple choice, checking all boxes that apply, or providing a constructed response). Scoring methods for such response data are well established.

By contrast, *process data* reflect interactions in which the student engages in and may provide relevant evidence about whether the student possesses a skill that is an assessment target. Thus, process data can be captured, measured, and interpreted to generate a score. Clickstream data, activity logs, text, and transcribed voice responses are among the ways to capture the state of student activity as they work through a problem. These types of data hold potential power to measure student interactivity in modeling and collaborative mathematics, as well as levels of any mathematical practice (e.g., capturing frequency, density, and intensity of engagement with a mathematical practice or identifying and comparing novice to expert levels of a practice through process data). While this capability is powerful in theory, moving from big data sources to carefully constructed and validated measures is difficult to achieve in practice. A special study in the area of mathematics assessment is needed to explore and fully realize the potential of process data within digital scenario-based tasks.

NAEP Mathematics Tools

The preceding sections provide an overview for thinking through—and developing—diverse ways to show what students know and can do mathematically. Each response type requires related system tools and, at times, mathematics tools. In a digitally based environment, for example, students will require tools to enter mathematical expressions; to draw, highlight, and erase on the screen; to measure the lengths of virtual objects; to plot points on number lines or in coordinate planes; to graph lines and functions; and to create and modify graphical representations. Additionally, the testing environment will need to provide computational tools equivalent to a four-function calculator at grade 4, a scientific calculator at grade 8, and a graphing calculator at grade 12. Continuing a practice that began with the 2017 NAEP Mathematics Assessment, before the assessment, students complete a brief interactive tutorial designed to orient them to the mathematics tools they will use during the assessment. The 2019 tutorials for each grade level can be found on the Internet (Governing Board, 2019a, 2019b).

The digitally based environment of the 2026 NAEP Mathematics Assessment provides the majority of these mathematics tools digitally. All digital NAEP assessments include system tools, which are always available and common across all NAEP assessments. There are also mathematics tools, which are specific to and only available for certain items on NAEP Mathematics Assessments. The materials and accompanying tasks need to be carefully chosen to cause minimal disruption of the administration process, and would typically only be provided when relevant to solving the item. Continuing the calculator policy established for the 2017 digital administration, students will have access to a calculator emulator in blocks of items designated as “calculator blocks.” New in 2026 will be the availability of a graphing emulator for grade 12, since high school students typically use graphing calculators or online emulators and not scientific calculators (Crowe & Ma, 2010).

Calculators

Calculator use has been recommended or mandated in high school mathematics in every U.S. state for more than 20 years, and research has explored the social, personal, civic, and economic consequences of such policies for nearly as long (see, e.g., Coiro, Knobel, Lankshear, & Leu, 2008; Voogt & Knezek, 2008). To date, most surveys of students and schools ask about types of calculators used, not about types of emulators or digital graphing environments. There is not yet a national data source on student access to graphing emulators. However, prevalence of use is indicated by the increasing use of textbooks at the high school level that include graphing emulator–embedded items in online homework problem sets and by the inclusion of graphing emulator items on state and multistate-consortium assessments (examples include the TI graphing calculator emulator on PARCC and Desmos software on SBAC).

New for the 2026 NAEP Mathematics Assessment at grade 12, “calculator” also refers to the use of a digital emulator for calculation and graphing, such as can be found on most state assessments. The assessment developer will propose additional restrictions on calculator use in grades 8 and 12, to (1) help ensure that items in calculator blocks cannot be solved in ways that are inconsistent with the knowledge and skills the items are intended to measure, and (2) maintain the security of NAEP test materials.

Allowance of a calculator during assessment administration should be taken into consideration when developing an item, so that the presence or absence of a calculator does not interfere with the measurement intent. For example, items assessing computational fluency should not allow for use of a calculator, as a calculator computation does not provide evidence of student computational skill (see Illustration 4.17a). In contrast, allowing for the use of a calculator when solving a multistep item in context can improve the reliability of the evidence of student knowledge and skills associated with the intended construct and avoid unintended assessment of a computational skill (see Exhibit 3.18).

On-Screen Mathematics Keyboard

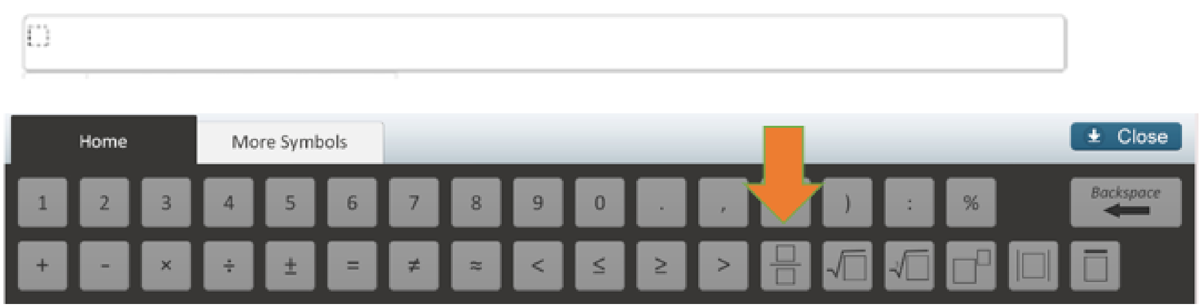
The item in Illustration 4.20 asks students to determine a probability and write their response as a fraction. The need to write the answer as a fraction allows for the use of the NAEP on-screen mathematics keyboard, which has a built-in functionality that allows students to choose a fraction shell and enter the numerators and denominators into response boxes within the fraction shell. FIB items that require a fractional answer or for which a common mathematical error could lead to a fractional answer should allow for use of the mathematics keyboard so that the determined answer can be entered without indicating the number type for the correct response. The on-screen mathematics keyboard available at each grade contains symbols appropriate for that grade, so not all symbols available at one grade are available at another. However, the fraction shell is located on the on-screen mathematics keyboard at all three grade levels.

Illustration 4.20. Short Constructed Response Example: On-Screen Mathematics Keyboard

Grade Level	Content Area	Assessed Practice(s)	Objective ID	Item Format
8	Data Analysis, Statistics, and Probability	Other	Data – 4.d	SCR – FIB

A standard number cube, numbered 1 through 6 on each side, is rolled three times. What is the probability of rolling a 2 on all three rolls? Express your answer as a fraction.

Enter your answer as a fraction in the space provided.



Scoring Information	
Key	1/216 or equivalent

The item in this illustration is based on a 2016 PARCC item with Item ID M20834, aligned to evidence statement 7.SP.8a.

Future Digital Tools

Examples of future digital mathematics tools for the 2026 NAEP Mathematics Assessment may include number tiles, spreadsheets, symbolic algebra manipulators, graphing tools, simulations, and dynamic geometry software. Continued development of mathematics tools (digital, physical, and other) can serve to achieve the goals of more authentic tasks for students and more diverse ways for students to demonstrate their knowledge and skills. Tools can allow for formal mathematics representations and symbols, and they can also allow students to create and share their own ways of thinking with their own representations. For example, some statistical tools allow students to construct their own graphical representations of data and create their own probability simulators. Considering what tools are needed for new items and the time it will take students to use them is an integral part of the assessment design process.

Accessibility

The mathematics assessments should be developed to allow for the participation of the widest possible range of students, so that interpretation of scores leads to valid inferences about levels of performance of the nation's students, as well as to valid comparisons across states. All students should have the opportunity to demonstrate their knowledge of the concepts and ideas that the NAEP Mathematics Assessment is intended to measure. To this end, item writing should follow the principles of universal design and sound testing practices as recommended by the National Center on Educational Outcomes (Thompson, Johnstone, Anderson, & Miller, 2005). These include attention to the population being assessed, precise definition of the constructs being assessed, review for fairness and accessibility of item content, clarity of the language and graphics used throughout the assessments, and the provision of accommodations without changing the constructs being assessed.

Although application of universal design principles to the item development process considers the ways in which the population being assessed can demonstrate learning, the use of such principles does not remove the need for accommodations altogether. With this in mind, items should be written to allow for necessary accommodations, including the use of online tools available to students during test administration, without changing the constructs being assessed.

The NAEP Mathematics Assessment is designed to measure student achievement across the nation. Consequently, NAEP incorporates inclusive policies and practices into every aspect of the assessment, including selection of students, participation in the assessment administration, and valid and effective accommodations. NAEP is administered to a sample of students who represent the student population of the nation, regardless of race/ethnicity, socioeconomic status, disability, status as an English language learner, or any other factors. Similarly, for state-level results and results for the NAEP Trial Urban District Assessment, NAEP is administered to a sample of students who represent the jurisdiction. Therefore, the NAEP Mathematics Assessment provides an opportunity for participating students to demonstrate mathematical knowledge and skill, including students who have learned mathematics in a variety of ways, followed different curricula, and used different instructional materials; students who have mastered mathematics content and practices to varying degrees; students with a variety of disabilities; and students who are English language learners. The related design issue is the development of a large-scale assessment that measures mathematics achievement of students who come to the assessment with different experiences, strengths, and challenges; who approach mathematics from different perspectives; and who have different ways of displaying their knowledge and skill.

NAEP uses two methods to design an accessible assessment program that provides accommodations for students with special needs. The first is addressed by careful item and delivery design with the full consideration of the range of participating students. For many students with disabilities and students whose native language is not English, the standard administration of the NAEP assessment will be most appropriate. For other students with disabilities (SD students) and some English language learners (ELL students), NAEP allows for a variety of accommodations, which can be used alone or in combination.

Some accommodations are built-in features, called Universal Design Elements, of the NAEP system tools that are available to all students. Other accommodations, such as additional assessment time, are offered for specific eligible students. Available accommodations fall into four categories:

- Standard NAEP Practice, available in almost all NAEP assessments for SD and ELL students.
- Other accommodations for SD students that require special presentation, such as Braille or sign language.
- Other accommodations for ELL students.
- Universal Design Elements that are built-in features of the computer-based assessments available to all students.

For more detailed information about accommodations, see the Governing Board's *NAEP Testing and Reporting of Students with Disabilities and English Language Learners Policy Statement* (2014b).

Matrix Sampling

The design of NAEP uses matrix sampling to enable a broad and deep assessment of students' mathematical knowledge and skill that also minimizes the time burden on schools and students. Matrix sampling is a sampling plan in which different samples of students take different samples of items. Students taking part in the assessment do not all receive the same items. Matrix sampling greatly increases the capacity to obtain information across a much broader range of the objectives than would otherwise be possible.

Balance of the Assessment

As mentioned earlier, the goal is to create an authentic assessment, one based on the experiences of students that will diversify the ways that students can show what they know and can do in mathematics. The emphasis placed on NAEP Mathematical Practices in the Framework increases interdependence since multiple practices may be assessed simultaneously in the context of one item. The expansion of item types to include scenario-based tasks also complicates the assessment design.

The balance of content and practices having been introduced in Chapters 2 and 3, respectively, a summary of all three balance dimensions follows.

- Balance by Mathematics Content
  - Number Properties and Operations
  - Measurement
  - Geometry
  - Data Analysis, Statistics, and Probability
  - Algebra

- Balance by Mathematical Practice
    - Representing
    - Abstracting and Generalizing
    - Justifying and Proving
    - Mathematical Modeling
    - Collaborative Mathematics
- Balance by Response Type
    - Selected response
    - Constructed response (short and extended)

Balance of Mathematics Content

Each NAEP Mathematics Assessment item or item part is developed to measure one content objective. Exhibit 4.2 reproduces the distribution of items by grade and content area (from Exhibit 2.1). See Chapter 2 for further details.

Exhibit 4.2. Percentage Distribution of Items by Grade and Content Area

<table>
<tr><th>Content Area</th><th>Grade 4</th><th>Grade 8</th><th>Grade 12</th></tr>
<tr><td>Number Properties and Operations</td><td>45*</td><td>20</td><td>10</td></tr>
<tr><td>Measurement</td><td>20</td><td>10</td><td rowspan="2">30</td></tr>
<tr><td>Geometry</td><td>15</td><td>20</td></tr>
<tr><td>Data Analysis, Statistics, and Probability</td><td>5</td><td>20</td><td>25</td></tr>
<tr><td>Algebra</td><td>15</td><td>30</td><td>35</td></tr>
</table>

* Note: At least one-third of grade 4 Number Properties and Operations items should assess fraction content.

Balance of Mathematical Practices

The target percentage ranges of items for each NAEP Mathematical Practice are reproduced in Exhibit 4.3 (from Exhibit 3.24). Most NAEP Mathematics Assessment items will feature one of the five NAEP Mathematical Practices (55 to 85 percent). The balance of items (15 to 45 percent), those in the "Other" category, will assess knowledge of content without calling on a particular NAEP Mathematical Practice. Because of the matrix sampling used on the NAEP Mathematics Assessment, the proportions in Exhibit 4.3 are for the entire pool of items used and do not represent the experience of each student. See Chapter 3 for further details about the NAEP Mathematical Practices.

Exhibit 4.3. Percentage Distribution of Items by NAEP Mathematical Practice

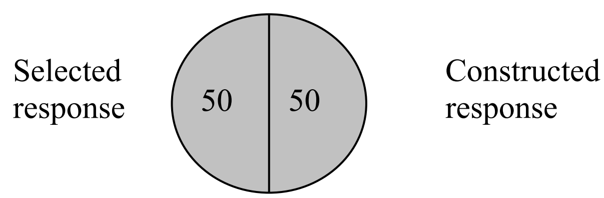
NAEP Mathematical Practice Area	Percentage of Items
Representing	10–15
Abstracting and Generalizing	10–15
Justifying and Proving	15–25
Mathematical Modeling	10–15
Collaborative Mathematics	10–15
Other	15–45

Certain formats are likely to be especially valuable in eliciting particular NAEP Mathematical Practices. As illustrated in Chapter 3, discrete items are useful measures of NAEP Mathematical Practices such as Representing, Abstracting and Generalizing, and Justifying and Proving. Also, as noted in Chapter 3, Mathematical Modeling and Collaborative Mathematics are more appropriately measured by scenario-based tasks.

Balance by Response Type

Items include selected response and constructed response types, and these response types may also occur within scenario-based tasks. Selected response includes traditional single-selection multiple choice, as well as other response types such as matching, zone, in-line choice, grid, and limited option responses. These items are machine scored. Constructed response includes short and extended constructed response. Constructed response items may include item types such as fill-in-the-blank, extended text, digital tool–based, and object-based constructed responses, as well as discourse and collaboration responses. Testing time on NAEP is divided evenly between selected response items and constructed response items, as shown in Exhibit 4.4.

Exhibit 4.4. Percent of Testing Time by Response Type



NAEP provides the nation with a snapshot of what U.S. students know and can do in mathematics. Results of the NAEP Mathematics Assessment administrations are reported in terms of average scores for groups of students on the NAEP 0–500 scale and as percentages of students who attain each of the three achievement levels (*NAEP Basic*, *NAEP Proficient*, and *NAEP Advanced*). This is an assessment of overall achievement, not a tool for diagnosing the needs of individuals or groups of students. Reported scores are always at the aggregate level; by law, scores are not produced for individual schools or students. Results are reported for the nation as a whole, for regions of the nation, for states, and for large districts that volunteer to participate in the NAEP Trial Urban District Assessment (TUDA). The NAEP results are published in an interactive version online as *The Nation's Report Card* (Governing Board, n.d.). The online resource provides detailed information on the nature of the assessment, the demographics of the students who participate, the assessment results, and the contexts in which students are learning.

Legislative Provisions for NAEP Reporting

Under the provisions of the Every Student Succeeds Act (ESSA), states receiving Title I grants must include assurance in their state plans that they will participate in the reading and mathematics state NAEP at grades 4 and 8. Local districts that receive Title I funds must agree to participate in biennial NAEP reading and mathematics administrations at grades 4 and 8 if they are selected to do so as part of the NAEP sample. Their results are included in state and national reporting. Participation in NAEP will not substitute for the mandated state-level assessments in reading and mathematics at grades 3 to 8. An important development over the last 20 years has been an evolving understanding of how NAEP complements state assessments, which are tightly aligned with state standards.

In 2002, NAEP initiated TUDA in five large urban school districts that are members of the Council of the Great City Schools (the Atlanta City, City of Chicago, Houston Independent, Los Angeles Unified, and New York City Public Schools districts). In 2003, additional large urban districts began to participate in these assessments, growing to a total of 27 districts by 2017. TUDA is administered biennially in odd-numbered years in tandem with NAEP state-level assessments. Sampled students in TUDA districts are assessed in the same subjects and use the same NAEP field materials as students selected as part of national main or state samples. TUDA results are reported separately from the state in which the TUDA is located, but results are not reported for individual students or schools. With student performance results reported by district, participating TUDA districts can use results for evaluating their achievement trends and for comparative purposes. Here too the complementarity of NAEP with state and local assessments is important to support so as to avoid unnecessary additional testing and to maximize useful information for educators and policymakers to use.

Reporting Scale Scores and Achievement Levels

The NAEP Mathematics Assessment is reported in terms of percentages of students who attain each of the three achievement levels: *NAEP Basic*, *NAEP Proficient*, and *NAEP Advanced*.

Reported scores are always at the aggregate level. The Framework calls for NAEP results to continue to be reported in terms of sub-scores as well, for each content area. Cut scores represent the minimum score required for performance at each NAEP achievement level. Cut scores are reported along with the percentage of students who scored at or above the cut score.

The Framework calls for reporting on NAEP Mathematical Practices. Since these practices are fundamentally intertwined with NAEP mathematics content areas, there will not be separate reporting scales for each NAEP Mathematical Practice. Options for measuring and reporting on NAEP Mathematical Practices are described in Appendix E.

Reporting on achievement levels is one way in which NAEP results reach the general public and policymakers. Since 1990, the Governing Board has used achievement levels for reporting results on NAEP assessments; achievement level results indicate the degree to which student performance meets the standards set for what students should know and be able to do at the *NAEP Basic, NAEP Proficient,* and *NAEP Advanced* levels. Descriptions of achievement levels articulate expectations of performance at each grade level (see Exhibit 5.1). They are reported as percentages of students within each achievement level range, as well as the percentage of students at or above *NAEP Basic* and at or above *NAEP Proficient* ranges. Students performing at or above the *NAEP Proficient* level on NAEP assessments demonstrate solid academic performance and competency over challenging subject matter.

It should be noted that the *NAEP Proficient* achievement level does not represent grade-level proficiency as determined by other assessment standards (e.g., state or district assessments) and there are significant differences between achievement in the context of NAEP as compared to the context of state-level annual tests. For one, teachers and students are not expected to have studied the NAEP framework or systematically aligned state standards or local curricula with it, nor are students expected to study intensively for the assessment. Furthermore, the NAEP assessment is broader than a typical state grade-level test, for NAEP covers multiple years of study and does not focus on specific instructional units and school years.

Results for students not reaching the *NAEP Basic* achievement level are reported as below *NAEP Basic*. As noted, individual student performance cannot be reported based on NAEP results.

NAEP Achievement Level Descriptions

Since 1990, the Governing Board has used achievement levels for reporting results on NAEP assessments. The achievement levels represent an informed judgment of "how good is good enough" in the various subjects that are assessed. Generic policy definitions for achievement at the *NAEP Basic, NAEP Proficient,* and *NAEP Advanced* levels describe in very general terms what students at each grade level should know and be able to do on the assessment. Achievement level descriptions specific to the 2026 NAEP Mathematics Framework can be found in Appendix A. These will be used to guide item development and initial stages of standard setting for the 2026 NAEP Mathematics Assessment, if it is necessary to conduct a new standard setting.

The content achievement level descriptions may be revised for achievement level setting, if additional information is obtained or required. A broadly representative panel of exceptional teachers, educators, and professionals in mathematics will be convened to engage in a standard-setting process to determine cut scores that correspond to the achievement level descriptions. All achievement level setting activities for NAEP are performed in accordance with current best practices in standard setting and the Governing Board's *Developing Student Achievement Levels for the National Assessment of Educational Progress Policy Statement* (2018a). The Governing Board policy does not extend to creating achievement level descriptions for performance below the *NAEP Basic* level.

Exhibit 5.1. Generic Achievement Level Policy Definitions for NAEP

Achievement Level	Definition
NAEP Advanced	This level signifies superior performance beyond *NAEP Proficient.*
NAEP Proficient	This level represents solid academic performance for each NAEP assessment. Students reaching this level have demonstrated competency over challenging subject matter, including subject-matter knowledge, application of such knowledge to real-world situations, and analytical skills appropriate to the subject matter.
NAEP Basic	This level denotes partial mastery of prerequisite knowledge and skills that are fundamental for performance at the *NAEP Proficient* level.

Contextual Variables

NAEP law (Governing Board, 2017b) requires reporting according to various student populations (see section 303[b][2][G]), including:

a. Gender,
b. Race/ethnicity,
c. Eligibility for free/reduced-price lunch,
d. Students with disabilities, and
e. English language learners.

At times, people presume that the categories used to report data are related to causal explanations for observed differences, for example, that gender accounts for performance. Although differences in student achievement are often referred to as "achievement gaps," scholars have long found that these differences also represent gaps in students' opportunities to learn (e.g., Carter & Welner, 2013; Flores, 2007; Martin, 2009; Schmidt et al., 2015), as discussed in Chapter 1. When results are interpreted in ways that emphasize achievement gaps without attending to opportunity gaps, score differences across subgroups of students can be misinterpreted as differences in student ability, rather than differences due to unequal and inadequate educational opportunities.

The *Standards for Educational and Psychological Testing* (AERA, APA, & NCME, 2014) recommend that reports of group differences in assessment performance be accompanied by relevant contextual information, where possible, to both discourage erroneous interpretation and enable meaningful analysis of the differences. That standard reads as follows:

> Reports of group differences in test performance should be accompanied by relevant contextual information, where possible, to enable meaningful interpretation of the differences. If appropriate contextual information is not available, users should be cautioned against misinterpretation. (Standard 13.6)

Contextual data about students, teachers, and schools are needed to fulfill the statutory requirement that NAEP include information, whenever feasible, for these groups which promotes meaningful interpretation. The important components of NAEP reporting are summarized in Exhibit 5.2.

Exhibit 5.2. Components of NAEP Reporting

Component	Key Characteristics
How Information Is Reported	Elements released to the public include: • Results published mainly online with an interactive report card • Dedicated website: Performance of various subgroups at the national level published online • Online data tools with sample questions, performance associated with all collected contextual variables, item maps, and profiles of states and TUDA districts
What Is Reported	NAEP data are reported by: • Percentage of students attaining achievement levels • Scale scores • Sample responses to illustrate achievement level definitions • Contextual information from NAEP questionnaires

Contextual variables are selected to be of topical interest, timely, and directly related to academic achievement and current trends and issues in mathematics. In the past, a range of information has been collected as part of NAEP. In one analysis, Pellegrino, Jones, and Mitchell (1999) identified five existing categories of indicators: (1) student background characteristics; (2) home and community support for learning; (3) instructional practices and learning resources; (4) teacher education and professional development; and (5) school climate.

Contextual variables for the 2026 NAEP Mathematics Assessment will build on two broad categories: student factors and opportunity to learn factors. Student factors have been described as skills, strategies, attitudes, and behaviors that are distinct from content knowledge and academic skills. Opportunity to learn factors have been described as whether students are exposed to opportunities to acquire relevant knowledge and skill in or out of school. These are described in the following section.

Mathematics-Specific Contextual Variables

As noted in Chapter 1, research has informed an expanded view of the factors that shape opportunities to learn, including time, content and practices, instructional strategies (e.g., how students are grouped for learning; the mathematical tasks they engage in; the opportunities students have to reason, model, and debate ideas), and instructional resources (e.g., human, material, and social resources that shape student access to mathematics).

For example, research has demonstrated that what students learn is shaped by the availability of various mathematics programs, curricula, extracurricular activities geared toward mathematics, the percentage of teachers certified in mathematics, teacher years of experience, percentage of mathematics teachers on an emergency license or vacancies/substitute teachers in the school, and number of teachers with mathematics degrees, among other factors. Teachers' and administrators' beliefs about what mathematics is, how one learns mathematics, and who can learn mathematics also affect student learning. What students learn is shaped by their sense of identity and agency. Students who see themselves, and who are seen by others, as capable mathematical thinkers are more likely to participate in ways that further their learning; students who do not see themselves, and are not seen by others, as capable mathematical thinkers are likely to be disengaged. Steele, Spencer, and Aronson (2002), for example, found that even passing reminders that a student is a member of one group or another—often, in this case, a group that is stereotyped as intellectually or academically inferior—can undermine student performance.

There are countless factors that shape what and when students learn. The NAEP Mathematics student, teacher, and administrator surveys cannot possibly cover all such factors. Even though it would be helpful to ask students and teachers the same questions, that too is not possible given time constraints. Furthermore, questions about some factors may not be appropriate in the NAEP context. Given the constraints, not all topics can be addressed.

To support prioritization and ensure that NAEP results have appropriate context for interpretation, the Framework sets the following topics to receive the greatest emphasis in the 2026 NAEP Mathematics Assessment's contextual questionnaires (in order of priority).

- *Mathematics content and practices*. The 2026 NAEP Mathematics Framework conceptualizes mathematics as both content and practices. Therefore, contextual variables related to mathematics content are expanded to include reference to NAEP Mathematical Practices as well. Interpreting students' achievement requires a basic understanding of what mathematics content and practices students have engaged with. Given variation across states in standards and frameworks, this information is crucial.
- *Teacher factors*. Research demonstrates that teacher quality is a critical in-school factor in predicting student achievement. The Framework prioritizes the collection of data on teacher preparation and professional development, as well as teacher mathematical knowledge for teaching.
- *Student mathematical identity*. Research demonstrates that students' perceptions of their mathematical identity directly relates to their mathematics learning. The 2026 NAEP Mathematics Framework prioritizes gathering information about students' mathematical identities through questions that address student participation in activities such as discussion of mathematical ideas or evaluation of how a mathematics problem is framed.

- *Instructional resources*. A range of resources influences instruction, including school climate, instructional leadership, additional instructional personnel, time, technology, curriculum, and materials. The Framework prioritizes gathering information about school resources that can inform the interpretation of results, including students' exposure to different types of technology, the time devoted to mathematics teaching and learning in school, and the curricular and instructional materials at teachers' and students' disposal to support learning. In terms of technology, questionnaires will emphasize what technology is *available* to support mathematics teaching and learning.
- *Instructional organization and strategies*. Interpreting student achievement levels will also depend on understanding the instructional strategies used in mathematics class, including collaborating in small-group work, engaging in mathematical discussions, and using a range of tools to represent and model mathematics. The Framework prioritizes gathering information both on the organization of classrooms and on the instructional routines and approaches that teachers use. It also includes what technologies and formative assessments are *used* in instruction.

Conclusion

As the Nation's Report Card, NAEP reports on student achievement over time, presenting an analysis of national trends in students' mathematical competence. The NAEP Mathematics Assessment is designed to assess the achievement of groups of students through robust and challenging assessments that are well aligned with current understanding of the mathematics content and practices to be learned and that use technology in ways that maximize both student engagement and accessibility. The results of the assessment are informed by data on contextual variables that illuminate potential differences in opportunities to learn for students.

Based on current research, policy, and practice, the NAEP Mathematics Framework visioning and development process articulated several major goals: to expand attention to student engagement in reasoning about and doing mathematics, to adjust NAEP's mathematics domains and competencies, to leverage interactive multimedia scenario-based tasks as a way to provide more authentic tasks for students to complete and to increase the assessment's accessibility, and to develop an expansive conception of opportunities to learn that would inform the collection and use of contextual information. Accordingly, Chapters 2 and 3 describe the content and practices of mathematics on which students should be measured on the 2026 NAEP Mathematics Assessment as the Nation's Report Card. Chapter 4 describes the expansion of the assessment in ways that prudently leverage technology's potential to increase authenticity and accessibility. Chapters 1 and 5 describe an expansive understanding of opportunities to learn, and the role that contextual information plays in meaningful interpretation of the results from future NAEP Mathematics Assessments based on the Framework.

The ultimate goal of our nation's schools is to ensure that every student has access to learning high-quality mathematics. NAEP plays an important role in providing a broad picture of students' knowledge and skills in mathematics to the nation. NAEP scores, illuminated by relevant contextual information, can provide the public, families, students, and schools useful data on student performance that complements information provided by state tests that are more tightly aligned with specific state standards. As a view of present trends, it provides invaluable data to inform policy and practice in the future.

Abstracting and Generalizing: A NAEP Mathematical Practice involving decontextualizing; identifying commonality across cases, items, problems, or representations; and extending one's reasoning to a broader domain appropriate for the grade level and the mathematics being assessed.

Achievement level descriptions (ALDs): Descriptions of student performance at achievement levels (basic, proficient, and advanced), detailing what students should know and be able to do in terms of the mathematics content areas and practices on the NAEP assessment.

Clickstream: Response and process data generated based on student interactions with tools and systems in scenario-based tasks.

Cognitive complexity: The state or quality of a thought process that involves numerous constructs, with many interrelationships among them. Such mental processing is often experienced as difficult or effortful.

Collaborative Mathematics: A NAEP Mathematical Practice that involves the social enterprise of doing mathematics with others through discussion and collaborative problem solving whereby ideas are offered, debated, connected, and built-upon toward solution and shared understanding. Collaborative mathematics involves joint thinking among individuals toward the construction of a problem solution.

Construct: An image, idea, or theory, especially a complex one formed from a number of simpler elements, and often embedded in a web of related ideas.

Constructed response: An open-ended, text-based response. Every constructed response item has a scoring guide that defines the criteria used to evaluate students' responses.

Context: The physical, temporal, historical, cultural, or linguistic setting for an event, performance, statement, or idea, and in terms of which such events or statements can be fully understood and assessed.

Contextual variables: Student, teacher, administrator, and school factors that shape students' opportunities to learn, including time, content, instructional strategies, and instructional resources.

Conversational response: A response within a discourse-based or collaborative task in which students respond by selecting from two or more choices that reflect a conversation between characters described in the task.

Deduction: Reasoning that makes a logical argument, draws conclusions, and applies generalizations to specific situations.

Discourse: Denotes written and spoken communications or "language-in-use" (Gee, 1999). Discourse can also refer to the totality of codified language used in a given field of intellectual enquiry and of social practice.

Discrete items: Stand-alone assessment items.

English language learner: Active learners of the English language who may benefit from various types of language support programs; students from a diverse set of backgrounds who often come from non-English-speaking homes and backgrounds, and who typically require specialized or modified instruction in both the English language and in their academic courses.

Funds of knowledge: The strengths students bring with them to the classroom, including academic and personal background knowledge, accumulated life experiences, skills and knowledge used to navigate everyday social contexts, and world views structured by broader historically and politically influenced social forces (Civil, 2016; González, Moll, & Amanti, 2005).

GAIMME: *Guidelines for Assessment and Instruction in Mathematical Modeling Education* (Garfunkel & Montgomery, 2019). A report issued by a collaboration between the Society for Industrial and Applied Mathematics and the Consortium for Mathematics and Its Applications.

Generalization: The act of identifying a property that holds for a larger set of mathematical objects or conditions than the number of individually verified cases.

Induction: Reasoning that begins with specific observations to develop generalizations and conclusions; looking for patterns and making generalizations.

In-line choice items: Items in which students respond by selecting one option from one or more drop-down menus that may appear in various sections of an item.

Instructional practice: Teaching methods that guide interaction in the classroom.

Joint thinking: Working and thinking together on a shared goal, including sharing ideas with others; attending to and making sense of the mathematical contributions of others; evaluating the merit of others' ideas through agreement or disagreement; and productively responding to others' ideas through building on or extending ideas and connecting or generalizing across ideas.

Justifying and Proving: A NAEP Mathematical Practice that involves creating, evaluating, showing, or refuting mathematical claims in developmentally and mathematically appropriate ways.

Mathematical argumentation: The action or process of reasoning systematically in support of an idea, action, or theory.

Mathematical justification: A critical aspect of the NAEP Mathematical Practice of Justifying and Proving that includes creating arguments, explaining why conjectures must be true or demonstrating that they are false, exploring special cases or searching for counterexamples, understanding the role of definitions and counterexamples, and evaluating arguments.

Mathematical knowledge for teaching: The specialized knowledge mathematics teachers need to support their students' learning that goes beyond the mathematics that any educated adult might need; the mathematics-specific knowledge of content, pedagogy, and students that is needed to perform the recurrent tasks of teaching mathematics to students (Ball, Thames, & Phelps, 2008).

Mathematical literacy: The application of numerical, spatial, or symbolic mathematical information to situations in a person's life as a community member, citizen, worker, or consumer.

Mathematical Modeling: A NAEP Mathematical Practice that involves making sense of a scenario, identifying a problem to be solved, mathematizing it, applying the mathematization to reach a solution, and checking the viability of the solution.

Mathematical practice: The working methods of *doing* mathematics, including the NAEP Mathematical Practices of Representing, Abstracting and Generalizing, Justifying and Proving, Mathematical Modeling, and Collaborative Mathematics.

Mathematical proof: A formal proof is a specific type of argument "consisting of logically rigorous deductions of conclusions from hypotheses" (NCTM, 2000, p. 55). The form used to represent a mathematical proof is valid as long as it communicates the essential features of the proof; that is, it contains logically connected mathematical statements that are based on valid definitions and theorems.

Mathematical problem solving: Completing mathematical tasks where the task contexts may range from the purely mathematical to those that are experientially concrete or real to students.

Mathematical reasoning: A skill that involves using other mathematical skills, including evaluating situations, selecting problem-solving strategies, drawing logical conclusions, developing and describing solutions, and recognizing how those solutions can be applied. Mathematical reasoners are able to reflect on solutions to problems and determine whether or not they make sense.

Object-based responses: Assessment responses that involve manipulating or using a physical object.

Opportunity gap: Relates to the inputs, the unequal or inequitable distribution of resources and opportunities, that contribute to and perpetuate lower educational achievement and attainment based on race, ethnicity, socioeconomic status, English proficiency, community wealth, familial situations, or other factors.

Opportunity to learn: Inputs and processes that enable student achievement of intended outcomes.

PISA: The Programme for International Student Assessment, an international assessment that measures 15-year-old students' reading, mathematics, and science literacy every three years.

Representing: A NAEP Mathematical Practice that involves recognizing, using, creating, interpreting, or translating among representations appropriate for the grade level and the mathematics being assessed.

Revoicing: A method of communication that can be used by students or teachers to "re-utter another's contribution through the use of repetition, expansion, or rephrasing" (Enyedy et al., 2008, p. 135).

Scenario-based task: Assessment tasks that have both context and extended storylines to provide opportunities to demonstrate facility with NAEP Mathematical Practices.

Selected response: Assessment responses that involve a student selecting one or more response options from a given, limited set of choices.

Single-selection multiple choice: Assessment items in which students respond by selecting a single choice from a set of given choices.

Student identity: A person's evolving view of self in a given social context influenced by their experiences, personal history, and other events. Students' mathematical identity is how they see themselves in relation to mathematics and mathematics learning (Bishop, 2012).

Tool-based responses: Assessment responses that involve manipulating or using a virtual tool on-screen (e.g., an on-screen ruler).

APPENDIX A: NAEP MATHEMATICS ACHIEVEMENT LEVEL DESCRIPTIONS

The NAEP Achievement Level Descriptions (ALDs) in this appendix provide examples of what students performing at the *NAEP Basic*, *NAEP Proficient*, and *NAEP Advanced* achievement levels should know and be able to do in terms of the mathematics content areas and practices identified in the Framework. The intended audiences for these ALDs are the NAEP assessment development contractor and item writers; the ALDs help ensure that a broad range of items is developed at each assessed grade.

The ALDs in the 2026 NAEP Mathematics Framework have changed, relative to ALDs presented in previous frameworks. The differences reflect not only changes to the mathematics knowledge, skills, and abilities assessed (mathematics content areas and mathematical practices) but also an effort to develop ALDs that provide explicit guidance for item developers. Specifically, across grade levels, the 2026 Framework ALDs have changed in the following ways:

- Updates to the grade-level objectives in Chapter 2 of the Framework are reflected in the content foci described in each grade-level ALD.
- Mathematical Practices are new to the 2026 Framework and are made explicit at every achievement level in every grade in these ALDs. The mathematical practices absorbed much of the reasoning and problem-solving language from previous framework ALDs. As noted in Chapter 3, some NAEP Mathematics items will not assess a NAEP Mathematical Practice. Thus, some elements of the NAEP Mathematics ALDs are not linked to a NAEP Mathematical Practice. Instead, they are associated with other activities such as enacting knowledge of mathematical facts, using procedural fluency, and engaging in mathematical practices that are not included in the five identified for the NAEP Mathematics Assessment.
- Although Chapter 4 of the Framework provides examples of digital tools (e.g., graphing tools) that may be common in 2026 and beyond in schools, these ALDs have reduced the focus on technology-specific descriptions of the mathematics students should know and be able to do on the NAEP Mathematics Assessment.
- To provide specific and unambiguous guidance to item developers, these ALDs provide more explicit elaborations of the knowledge and skills students should demonstrate and the actions they should perform at each grade level and within each achievement level.

Within each grade level, the shifts from one achievement level to the next have commonalities, and the content of each achievement level can be described generally. Descriptions at each achievement level, for all grade levels, are as follows:

- Descriptions at the *NAEP Basic* level focus on emerging understanding of grade-appropriate concepts and introductory engagement with mathematical practices.
- Descriptions at the *NAEP Proficient* level focus on application of grade-appropriate concepts and skillful engagement with mathematical practices.
- Descriptions at the *NAEP Advanced* level focus on extension of grade-appropriate concepts and expert engagement with mathematical practices.

Text that elaborates on these statements is included within the ALD tables.

To add clarity and specificity, the 2026 Framework ALDs include example items targeting each achievement level within each grade level. Following the ALDs presentation, in Appendix B, three sets of items (one set each for grades 4, 8, and 12) illustrate the knowledge and skills required at different NAEP achievement levels. The items are not intended to represent the entire set of mathematics content areas or practices, nor do the items imply priority or importance of some content areas or practices above others.

Finally, to guard against misinterpretations, it is important to clarify the intended meaning of the term *routine*, which is used frequently in the ALDs. For the purposes of the ALDs, *routine* is defined as *having a readily available solution method*.

Mathematics Achievement Level Descriptions for Grade 4

NAEP Basic	**Grade 4 students performing at the *NAEP Basic* level should show evidence of emergent understanding of mathematics concepts and procedures in the five NAEP content areas. Students should also show evidence of engagement in the five NAEP Mathematical Practices as detailed.** Grade 4 students performing at the *NAEP Basic* level should be able to estimate and perform paper and pencil computations with whole numbers (e.g., addition and subtraction within 1,000; multiplication and division within 100); understand the meaning of fractions and decimals, but not necessarily the relations between fractions and decimals; compare numbers to familiar benchmarks such as 0, ¼, ½, ⅔, ¾, and 1; identify or measure attributes of simple plane figures (e.g., triangles, rectangles, squares, and circles) and simple solid figures (e.g., cubes, spheres, and cylinders), choosing appropriate measuring tools and units of measure; and solve problems involving these concepts and procedures. Students should be able to represent whole numbers, fractions, and decimals using visual representations; draw or sketch simple plane figures from a written description; create a visual, graphical, or tabular representation of a given set of data; and recognize, describe (in words or symbols), or extend numerical and visual patterns. They should be able to explain or defend strategies or solutions (e.g., justify solutions to word problems through numeric representations and operations); make mathematical sense of a problem scenario; select and use visual, physical, or symbolic representations, as needed, to lead to solutions; and share ideas and revoice the ideas of others.
NAEP Proficient	**Grade 4 students performing at the *NAEP Proficient* level should be able to recognize when particular concepts, procedures, and strategies are appropriate, and select, integrate, and apply them to represent or model situations mathematically and solve problems requiring more than the application of a known procedure or strategy. Students should be able to reason about relationships involving the domains of number, space, or data. Students should also show evidence of engagement in the five NAEP Mathematical Practices as detailed.** Grade 4 students performing at the *NAEP Proficient* level should be able to estimate and compute with whole numbers (within the guidelines set by the NAEP objectives) and determine whether and explain why the results are reasonable; identify, represent, compare, add, and subtract fractions and decimals, using visual representations to compare numbers and as tools to solve problems; identify or draw angles; draw or sketch simple plane and solid figures from a written description; read and interpret a single set of data, including the interpretation of graphical or tabular representations of data; extend their understanding of patterns to create a different

	representation of a pattern or sequence; and create, use, and defend visual representations of problem situations involving these concepts and procedures. In all content areas, students should be able to abstract or de-contextualize and re-contextualize ideas in routine problems using written and symbolic structures; create and evaluate mathematical arguments; explain why conjectures must be true or demonstrate that they are false; explore with examples or search for counterexamples and understand the role of counterexamples in mathematical arguments; determine assumptions, pose answerable questions, and determine tools to use as they interpret and solve problems; and make sense of and evaluate the mathematical contributions of others through expressing and defending agreement or disagreement.
NAEP Advanced	**Grade 4 students performing at the *NAEP Advanced* level should be able to apply conceptual understanding and procedural knowledge in non-algorithmic ways to complex and non-routine mathematical or real-world problems in the five NAEP content areas. Students should also show evidence of engagement in the five NAEP Mathematical Practices as detailed.** Grade 4 students performing at the *NAEP Advanced* level should be able to solve complex and non-routine real-world problems in all NAEP content areas. These students should be able to draw logical conclusions from the results of a solution process; justify answers and solution processes by explaining how and why they were achieved; and use words or symbols to generalize a pattern appearing in a sequence or table. Students should be able to build on, analyze, and justify representations or mathematical models created by others; use structures and patterns to generate a rule and investigate conditions under which the rule applies; use a variety of grade-appropriate methods to justify or refute a mathematical statement using valid definitions, statements, or counterexamples; determine and use a series of processes to mathematize a complex or non-routine situation and evaluate the results obtained; and extend, connect, or generalize across the ideas of others.

Mathematics Achievement Level Descriptions for Grade 8

NAEP Basic	**Grade 8 students performing at the *NAEP Basic* level should show evidence of emergent understanding, recognition, and application of concepts and procedures in the five NAEP content areas. Students should show evidence of engagement in the five NAEP Mathematical Practices as detailed.** Grade 8 students performing at the *NAEP Basic* level should be able to estimate and perform paper-and-pencil computations with rational numbers, including integers; solve linear equations or inequalities; choose appropriate measuring tools and units of measure; and solve problems involving strategic reasoning with these concepts and procedures, including using proportional reasoning to represent and solve routine problems. Students should be able to visually represent rational numbers, including decimals and integers, and use these representations as tools to solve problems; draw or sketch polygons, circles, or semicircles from a written description; create a visual, graphical, or tabular representation of a given set of data; and recognize, describe (in words or symbols), or extend numerical and visual patterns. They should be able to explain or defend strategies or solutions (e.g., justify solutions to word problems through numeric representations and operations); make mathematical sense of a problem scenario, selecting and using visual, physical, or symbolic representations, as needed, to lead to solutions; and share ideas and revoice the ideas of others.
NAEP Proficient	**Grade 8 students performing at the *NAEP Proficient* level should show evidence of recognizing and applying concepts and procedures to solve problems requiring more than routine application of a known process or result in the five NAEP content areas. They should recognize when particular concepts, procedures, and strategies are appropriate and select, integrate, and apply them to represent or model situations mathematically. Students should be able to reason about relationships involving the domains of number, space, or data. Students should also show evidence of engagement in the five NAEP Mathematical Practices as detailed.** Grade 8 students performing at the *NAEP Proficient* level should understand the connections among integers, fractions, percents, and decimals and be able to work across these sets of numbers to examine proportional and linear relationships; expand their understanding of algebraic relationships to translate between different representations, compare properties of two relationships each represented differently, identify linear functions, and use the structure of an algebraic expression to solve problems; estimate the size of an object with respect to a given measurement attribute (e.g., length, area, volume, angle measurement, weight, or mass); compare figures or objects with respect to a measurement attribute; identify, describe, and justify relationships of congruence, similarity, and symmetry; organize data in order to make inferences and draw conclusions, interpret data in terms of generalized

	phenomena (e.g., shape, center, spread, clusters), and make comparisons or explore differences within and among sets of data; and interpret and apply probability concepts to routine situations. In all content areas, students should be able to abstract or de-contextualize and re-contextualize ideas in routine problems using written and symbolic structures; create and evaluate mathematical arguments; explain why conjectures must be true or demonstrate that they are false; explore with examples or search for counterexamples and understand the role of definitions and counterexamples in mathematical arguments; determine assumptions, pose answerable questions, and determine tools to use as they interpret and solve problems; and make sense of and evaluate the mathematical contributions of others through expressing and defending agreement or disagreement.
NAEP Advanced	**Grade 8 students performing at the *NAEP Advanced* level should be able to apply conceptual understanding and procedural knowledge in non-algorithmic ways to complex and non-routine mathematical or real-world problems. They should also be able to justify, generalize, and apply concepts and procedures, and be able to synthesize concepts and processes in the five NAEP content areas. Students should also show evidence of engagement in the five NAEP Mathematical Practices as detailed.** Grade 8 students performing at the *NAEP Advanced* level should be able to solve complex and non-routine real-world problems in all NAEP content areas. They should be able to probe examples and counterexamples in order to shape generalizations from which they can develop mathematical models; use number sense and geometric awareness (e.g., definitions, properties of and relationships between geometric figures, results of transformations) to consider the reasonableness of an answer; and create problem-solving techniques, explaining the reasoning processes underlying their conclusions. Students should be able to use, analyze, and justify representations created by others; use structures and patterns to generate a rule and investigate conditions under which the rule applies; use a variety of grade-appropriate proof methods to justify a mathematical statement using valid definitions, statements, or counterexamples; determine and use a series of processes to mathematize a complex or non-routine situation and evaluate the results obtained; and extend, connect, or generalize across the ideas of others.

Mathematics Achievement Level Descriptions for Grade 12

NAEP Basic	**Grade 12 students performing at the *NAEP Basic* level should show evidence of emergent understanding, recognition, and application of concepts and procedures in the five NAEP content areas. Students should also show evidence of engagement in the five NAEP Mathematical Practices as detailed.** Grade 12 students performing at the *NAEP Basic* level should be able to estimate and perform computations with real numbers, including irrational numbers; select appropriate units related to representing or measuring an attribute of an object; identify and describe relationships of congruence, similarity, and symmetry; organize data in order to make inferences and draw conclusions; interpret data in terms of generalized phenomena (e.g., shape, center, spread, clusters); make comparisons or explore differences within and among sets of data; interpret and apply probability concepts to routine situations; recognize, identify, and interpret information about functions presented in various forms; and solve problems involving these concepts and procedures, including using the coordinate plane to model and solve routine problems. Students should be able to represent real numbers, including very large and very small numbers, using visual representations and numerical expressions (e.g., scientific notation), and use these representations and expressions as tools to solve problems; draw or sketch plane figures and planar images of three-dimensional figures from a written description; create a visual, graphical, or tabular representation of a given set of data; and recognize, describe, or extend numerical patterns, including arithmetic and geometric progressions. They should be able to explain or defend strategies or solutions (e.g., justify solutions to word problems through numeric representations and operations); make mathematical sense of a problem scenario, selecting and using visual, physical, or symbolic representations, as needed, to lead to solutions; and share ideas and revoice the ideas of others.
NAEP Proficient	**Grade 12 students performing at the *NAEP Proficient* level should be able to recognize when particular concepts, procedures, and strategies are appropriate and to select, integrate, and apply them to represent or model situations mathematically to solve problems requiring more than the application of a known result. Students should be able to reason about relationships involving the domains of number, space, or data. Students should also show evidence of engagement in the five NAEP Mathematical Practices as detailed.** Grade 12 students performing at the *NAEP Proficient* level should be able to solve complex non-routine items using algebraic and geometric approaches. Students should be able to find, test, and validate geometric and algebraic results and conjectures using a variety of methods. They should be able to

	design and carry out statistical surveys and experiments and interpret results that are obtained by them or by others. Students should also be able to translate between representations of functions (linear and nonlinear, quadratic and exponential), including verbal, graphical, tabular, and symbolic representations. In all content areas, students should be able to abstract or de-contextualize and re-contextualize ideas in routine problems using written and symbolic structures; create and evaluate mathematical arguments; explain why conjectures must be true or demonstrate that they are false; explore with examples or search for counterexamples and understand the role of definitions and counterexamples in mathematical arguments; determine assumptions, pose answerable questions, and determine tools to use as they interpret and solve problems; and make sense of and evaluate the mathematical contributions of others through expressing and defending agreement or disagreement.
NAEP Advanced	**Grade 12 students performing at the *NAEP Advanced* level should demonstrate in-depth knowledge of and be able to reason about mathematical concepts and procedures in the realms of number, algebra, geometry, and statistics. Students should also show evidence of engagement in the five NAEP Mathematical Practices as detailed.** Grade 12 students performing at the *NAEP Advanced* level should be able to defend their solutions to complex non-routine tasks. Students should be able to reason about and with functions and transformations, using properties of functions and transformations to analyze relationships and to determine and construct appropriate representations for solving problems; explain or defend reasoning processes; and understand the role of hypotheses, deductive reasoning, and conclusions in geometric proofs and algebraic arguments made by themselves and others. Students should be able to use, analyze, and justify representations created by others; use structures and patterns to generate rules and investigate the conditions under which rules apply; use a variety of grade-appropriate proof methods to justify a mathematical statement using valid definitions, statements, theorems, or counterexamples; determine and use a series of processes to mathematize a complex or non-routine situation and evaluate the results obtained; and extend, connect or generalize across the ideas of others.

NAEP Basic, NAEP Proficient, and NAEP Advanced Achievement Levels for Grade 4

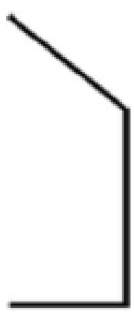
NAEP Basic, Grade 4

In this item, students are given an incomplete representation of a shape and asked to identify an associated complete shape, addressing *NAEP Basic* level language "identify or measure attributes of simple plane figures."

Subject: Mathematics, Grade: 4, Year: 2013
Content Classifications: Geometry, Low, Type: MC, Difficulty Level: Easy

The correct answer is:

A. Pentagon



Part of a closed shape is shown above. When the shape is completed, which of these could it be?

A. Pentagon

B. Rectangle

C. Square

D. Triangle

NAEP Proficient, Grade 4

In this item, students are presented with a problem situation involving multistep computation and interpretation within the context of the situation, addressing *NAEP Proficient* level language "estimate and compute with whole numbers (within the guidelines set by the NAEP objectives)" and "abstract or de-contextualize and re-contextualize ideas in routine problems."

Subject: Mathematics, Grade: 4, Year: 2017

Content Classifications: Number properties and operations, Moderate, Type: SR, Difficulty Level: Hard

A school will receive between \$600 and \$900 to spend on art supplies.

The money will be given to three school clubs.

Each school club will get the same amount of money.

Which of the following amounts of money could each school club get?

Select all the correct answers.

- A ☐ \$145
- B ☐ \$225
- C ☐ \$295
- D ☐ \$325
- E ☐ \$355

Clear Answer

NAEP Advanced, Grade 4

In this item, students are presented with a specific mathematical scenario and asked to generalize the results and provide a justification for the generalization, addressing *NAEP Advanced* level language “use structures and patterns to generate a rule” and “use a variety of grade-appropriate methods to justify or refute a mathematical statement [the rule] using valid definitions, statements, or counterexamples.”

Subject: Mathematics, Grade: 4, Year: 2011
Content Classifications: Number properties and operations, High, Type: SCR, Difficulty Level: Hard

Mr. Jones picked a number greater than 100.

He told Gloria to divide the number by 18.

He told Edward to divide the number by 15.

Whose answer is greater?

○ Gloria's ○ Edward's

Explain how you know this person's answer will always be greater for any number that Mr. Jones picks.

NAEP Basic, NAEP Proficient, and NAEP Advanced Achievement Levels for Grade 8

For each of items 1 through 4, refer to the following three figures.

Figure 1

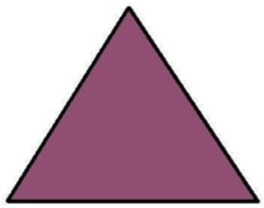


Figure 2

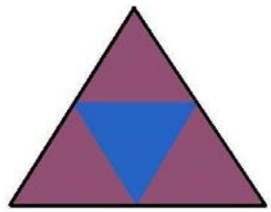
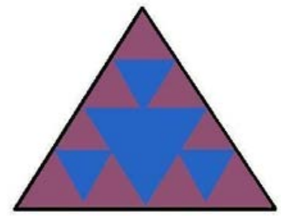


Figure 3



NAEP Basic, Grade 8

Item 1.

Figure 1 is an equilateral triangle and s is the length of a side of the triangle. P is the perimeter of the triangle in Figure 1. Complete the equation for the perimeter, P, of Figure 1.

$$P = \boxed{} \cdot s$$

This item is an indicator of *NAEP Basic* because students are asked to recognize or apply directly procedures and representations that are routine at grade 8 regarding perimeter of triangles.

Item 2.

In Figure 2 the blue triangle has been created by connecting the midpoints of the sides of the original triangle in Figure 1. Indicate if each of the following statements is true or false:

a) The perimeter of the blue triangle is one-fourth the perimeter of the original triangle
b) The perimeter of the blue triangle is one-half the perimeter of the original triangle
c) The area of the blue triangle is one-fourth the area of the original triangle
d) The area of the blue triangle is one-half the area of the original triangle

This item is an indicator of *NAEP Basic* because students are asked to recognize or apply simple relationships regarding area and perimeter of triangles.

NAEP Proficient, Grade 8

Item 3.

> Figure 1 is an equilateral triangle, and s is the length of a side of the triangle. In Figure 2 the blue triangle has been created by connecting the midpoints of the sides of the original triangle. In Figure 3 the smaller blue triangles have been created by connecting the midpoints of the sides of each interior triangle in Figure 2.
>
> 1) Express the perimeter of the blue triangle in Figure 2 in terms of s.
> 2) Express the sum of the perimeters of all the blue triangles in Figure 3 in terms of s.

Item 3 is an indicator of *NAEP Proficient* because it involves applying a well-known procedure to solve a non-routine problem that should be accessible to grade 8 students and representing the solution using grade-appropriate algebraic representations.

NAEP Advanced, Grade 8

Item 4.

> Figure 1 is an equilateral triangle. In Figure 2 the blue triangle has been created by connecting the midpoints of the sides of the original triangle. In Figure 3 the smaller blue triangles have been created by connecting the midpoints of the sides of each interior triangle in Figure 2. Suppose you continue this process of connecting midpoints to obtain subsequent figures (Figure 4, Figure 5, Figure 6, and so on).
>
> 1) Express the sum of the perimeters of all the blue triangles in Figure 5 in terms of s.
> 2) Express the sum of the perimeters of all the blue triangles in Figure 10 in terms of s.

Item 4 is an indicator of *NAEP Advanced* because it involves generalizing a pattern and using a well-known procedure in the context of the pattern to solve a non-routine problem, and representing the solution using grade-appropriate algebraic representations.

NAEP Basic, NAEP Proficient, and NAEP Advanced Achievement Levels for Grade 12

NAEP Basic, Grade 12

In this item, students are given pairs of shapes and asked to identify the pair that must always be similar, addressing *NAEP Basic* level language "identify and describe relationships of congruence, similarity, and symmetry."

Subject: Mathematics, Grade: 12, Year: 2005
Content Classifications: Geometry, Low, Type: MC, Difficulty Level: Medium

The correct answer is:

A. Two equilateral triangles

Which of the following pairs of geometric figures must be similar to each other?

A. Two equilateral triangles

B. Two isosceles triangles

C. Two right triangles

D. Two rectangles

E. Two parallelograms

NAEP Proficient, Grade 12

In this item, students are asked to select the data collection method most appropriate for the question of interest, addressing *NAEP Proficient* level language "They should be able to design and carry out statistical surveys."

Subject: Mathematics, Grade: 12, Year: 2009
Content Classifications: Data analysis, Statistics, and Probability, Moderate, Type: MC, Difficulty Level: Medium

The correct answer is:

C. Randomly select 25 students from a list of all students at the school.

The principal of a high school would like to determine why there has been a large decline during the year in the number of students who buy food in the school's cafeteria. To do this, 25 students from the school will be surveyed. Which method would be the most appropriate for selecting the 25 students to participate in the survey?

A. Randomly select 25 students from the senior class.

B. Randomly select 25 students from those taking physics.

C. Randomly select 25 students from a list of all students at the school.

D. Randomly select 25 students from a list of students who eat in the cafeteria.

E. Give the survey to the first 25 students to arrive at school in the morning.

NAEP Advanced, Grade 12

In this item, students need to use geometric properties, definitions, and principles to describe a geometric process for finding the center of any circle, addressing *NAEP Advanced* level language "use a variety of grade-appropriate proof methods to justify a mathematical statement using valid definitions, statements, theorems, or counterexamples."

Subject: Mathematics, Grade: 12, Year: 1996
Content Classifications: Geometry, Problem solving, Type: ECR, Difficulty Level: Hard

This question requires you to show your work and explain your reasoning. You may use drawings, words, and numbers in your explanation. Your answer should be clear enough so that another person could read it and understand your thinking. It is important to show <u>all</u> your work.

Describe a procedure for locating the point that is the center of a circular paper disk. Use geometric definitions, properties, or principles to explain why your procedure is correct. Use the disk provided to help you formulate your procedure. You may write on it or fold it any way that you find helpful, but it will not be collected.

The Visioning Panel was tasked with formulating "high-level guidance about the state of the field to inform the process." The specific charge stated:

> The Visioning and Development Panels will recommend to the Governing Board how best to balance necessary changes in the NAEP Mathematics Framework at grades 4, 8, and 12 with the Governing Board's desire for stable reporting of student achievement trends and assessment of a broad range of knowledge and skills, so as to maximize the value of NAEP to the nation; and the Panels are also tasked with considering opportunities to extend the depth of measurement and reporting given the affordances of digitally based assessment.

The 30-person Visioning Panel met in November 2018 to determine principles, goals, and policies to guide the NAEP Mathematics Framework update. During this meeting, the Visioning Panel learned about NAEP, the framework update process, and available NCES resources. Using this information, panelists identified and discussed issues related to developments in mathematics education research, policy, and practice that should inform the design of the assessment framework. The Visioning Panel then developed guidelines for recommended updates. The guidelines were clustered in three domains: mathematics, assessment design and technology, and opportunities to learn.

The full set of guidelines was passed on to the Development Panel, fifteen Visioning Panelists who were tasked with developing drafts of updated project documents and engaging in deliberations about how issues outlined in the guidelines should be reflected in the framework. The project documents include: a recommended framework, assessment and item specifications, and recommendations for contextual variables that relate to the subject being assessed. The Development Panel convened four 2-day meetings to prepare these three documents, as well as webinars to prepare for and review progress. In between and after meetings, the Development Panel drafted and revised documents. The updates included responding to the guidelines set by the Visioning Panel. These guidelines are summarized on the following pages.

MATHEMATICS

1. EXPANSION OF ATTENTION TO STUDENT REASONING AND MATHEMATICAL PRACTICES

We recommend defining mathematical practice constructs of priority interest in the framework (e.g., representing, abstracting and generalizing, justifying and proving, modeling, mathematical collaboration), providing examples of how they can be assessed (e.g., in the Assessment and Item Specifications), and using these definitions to systematically assess these practices, integrated with content, in 2026.

2. SIGNIFICANT BROADENING OF MATHEMATICAL DOMAINS AND COMPETENCIES

The mathematics content of the preK–12 curriculum has significantly evolved, and these changes need to be reflected in NAEP. **We recommend** a broadening of the content in several ways, including:

- (a) content that reflects research on mathematics teaching and learning that responds to students' diverse experiences, backgrounds, language, and culture;
- (b) a re-examination of statistics, data analysis and probability concepts and skills in light of current scholarship and standards documents;
- (c) attention to a wider range of technological tools available for students;
- (d) highlighting foundational mathematical themes that cut across different areas of content domains (e.g., geometry, algebra) and the grade bands from grades 4 to 8 to 12; and
- (e) consideration of a new cross-cutting theme or content area (at grade 12) that expands on calculus-readiness and statistics to include increasingly relevant applied mathematics important to informed citizenship, to personal financial and other decisions, and a variety of careers.

3. ATTENTION TO THE BALANCE OF COGNITIVE DEMAND

NAEP's current levels of "mathematical complexity" afford a balance between low-level items that ask for recall or demonstration of procedures, medium-level items that require connection-making on multistep procedures, and high-level items that require analysis, creativity, synthesis, or justification and proof. **We recommend** a NAEP mathematics framework update in terms of relevant research on mathematical complexity and cognitive demand.

TEST DESIGN AND TECHNOLOGY

4. TEST DESIGN

We recommend the integration of content and practice skills through leveraging interactive multimedia scenario-based tasks as a way to provide more authentic tasks for students to complete (e.g., NAEP Technology and Engineering Literacy; see online TEL tasks).

5. STRATEGIC USE OF TECHNOLOGY

We recommend that NAEP revisions leverage technology to increase the assessment's authenticity (allowing students to use the technologies they use in and out of school) and the assessment’s accessibility. Given the digital divide, as the NAEP instrument evolves, panels should address known and potential implementation issues and recommend ways to mitigate issues of access and test-taking that could occur in under-resourced communities.

OPPORTUNITIES TO LEARN AND OPPORTUNITIES TO DEMONSTRATE LEARNING

6. EXPANSIVE CONCEPTION OF OPPORTUNITIES TO LEARN
We recommend developing a broad approach to the framework update that scaffolds attention to opportunities to learn mathematics content, processes, and practices. This intent should be woven into the objectives in the framework, the item types and examples, and realized in contextual variables used on surveys.

We recommend updates to contextual variables in surveys that include attention to students’ views of mathematics, and of themselves as mathematics learners; students’ views of their peers’, teachers’, and school’s beliefs/interest in their progress in mathematics; students’ views of mathematics teaching and mathematics assessment (including NAEP); student access to and engagement with the language and culture of the test; teachers’ knowledge of what has been taught before NAEP is administered; and teachers’ beliefs about mathematics, mathematics teaching, and what their students can do.

7. ACCESSIBLE ASSESSMENTS FOR ALL STUDENTS
We recommend developing authentic assessment items with multiple access points that provide diverse populations of students with opportunities to demonstrate their mathematical knowing and reasoning in creative, authentic ways. This includes improving the accessibility of the assessment through short term goals like reconsidering test time limits, establish testing conditions that are more closely aligned with learning conditions (the use of typical tools, for example, or allowing teachers to be present) as well as longer term efforts to document how the current assessment remains inaccessible. Items should have consequential validity, be engaging to students, reflect guidelines for “low floor, high ceiling” tasks that provide opportunities for multiple approaches, and connect to students’ lived experiences and funds of knowledge. Making the testing technologies widely available to students and teachers well before the assessment would also increase access and authenticity. Finally, because some research suggests that using mathematics tasks situated in everyday situations allows students to bring greater meaning to those tasks, we believe the authenticity of assessment items may allow for a more successful assessment of the mathematics students are learning (Boaler, 2002; Tomaz & David, 2015).

To assist item writers when coordinating decisions about item content and NAEP Mathematical Practices, the tables in this appendix include the objectives from Chapter 2 and, in an additional column, examples of the NAEP Mathematical Practice(s) likely to be assessed by each objective. The assignment of NAEP Mathematical Practices to objectives is based on the verb usage in the objectives and their alignment to the description of a particular practice or practices.

The listed "Inherent Practice(s)" are not definitive, nor are they intended as a boundary on how an objective and a practice might be assessed together. See Chapter 3 for detailed discussion and illustration of the NAEP Mathematical Practices.

Some cells in the "Inherent Practice(s) column" do not name a NAEP Mathematical Practice, and instead contain "Other" or "Variable."

- Other: The content of this objective lends itself to recall, procedural fluency, or mathematical practice(s) other than NAEP Mathematical Practices.
- Variable: More than one practice is likely to work well with the objective. The direction taken with the development of context, activity, prompts, or questions in an item aligned to the listed content objective would determine which, if any, of the NAEP Mathematical Practices is also assessed.

As in Chapter 2, the symbol * is used to identify mathematics content beyond that typically taught in a standard 3-year course of study (the equivalent of 1 year of geometry and 2 years of algebra with statistics) and # is used to indicate opportunities for a mathematical literacy focus.

Number Properties and Operations

Num – 1. Number sense			
Grade 4	**Grade 8**	**Grade 12**	**Inherent Practice(s)**
a) Identify place value and actual value of digits in whole numbers, and think flexibly about place value notions (e.g., there are 2 hundreds in 253, there are 25 tens in 253, there are 253 ones in 253).	a) Use place value to represent and describe integers and decimals.		Representing
b) Represent numbers using base 10, number line, and other representations.	b) Represent or describe rational numbers or numerical relationships using number lines and diagrams.		Representing

Num – 1. Number sense (continued)			
Grade 4	**Grade 8**	**Grade 12**	**Inherent Practice(s)**
c) Compose or decompose whole quantities either by place value (e.g., write whole numbers in expanded notation using place value: 342 = 300 + 40 + 2 or 3 × 100 + 4 × 10 + 2 × 1) or convenience (e.g., to compute 4 × 27 decompose 27 into 25 + 2 because 4 × 25 is 100, and 4 × 2 is 8 so 4 × 27 is 108).			Representing; Abstracting and Generalizing
d) Write or rename whole numbers (e.g., 10: 5 + 5, 12 – 2, 2 × 5).	d) Write or rename rational numbers.	# d) Represent, interpret, or compare expressions for real numbers, including expressions using exponents and *logarithms.	Representing
e) Connect across various representations for whole numbers, fractions, and decimals (e.g., number word, number symbol, visual representations).	e) Recognize, translate, or apply multiple representations of rational numbers (fractions, decimals, and percents) in meaningful contexts.		Representing; Abstracting and Generalizing
	f) Express or interpret large numbers using scientific notation from real-life contexts.	# f) Represent or interpret expressions involving very large or very small numbers in scientific notation.	Representing
	g) Find absolute values or apply them to problem situations.	g) Represent, interpret, or compare expressions or problem situations involving absolute values.	Representing
h) Recognize and generate simple equivalent (equal) fractions and explain why they are equivalent (e.g., by using drawings).	h) Order or compare rational numbers (fractions, decimals, percents, or integers) using various representations (e.g., number line).		Representing
i) Order or compare whole numbers, decimals, or fractions using common denominators or benchmarks.	i) Order or compare rational numbers including very large and small integers, and decimals and fractions close to zero.	# i) Order or compare rational or irrational numbers, including very large and very small real numbers.	Other

Num – 2. Estimation			
Grade 4	**Grade 8**	**Grade 12**	**Inherent Practice(s)**
a) Use benchmarks (well-known numbers used as meaningful points for comparison) for whole numbers, decimals, or fractions in contexts (e.g., ½ and 0.5 may be used as benchmarks for fractions and decimals between 0 and 1.00).	a) Establish or apply benchmarks for rational numbers and common irrational numbers (e.g., π) in contexts.		Variable
b) Make estimates appropriate to a given situation with whole numbers, fractions, or decimals.	b) Make estimates appropriate to a given situation by: • Identifying when estimation is appropriate, • Determining the level of accuracy needed, • Selecting the appropriate method of estimation.	# b) Identify situations where estimation is appropriate, determine the needed degree of accuracy, and *analyze the effect of the estimation method on the accuracy of results.	Variable
c) Verify and defend solutions or determine the reasonableness of results in meaningful contexts.	c) Verify solutions or determine the reasonableness of results in a variety of situations, including calculator or computer results.	# c) Verify solutions or determine the reasonableness of results in a variety of situations.	Representing
	d) Estimate square or cube roots of numbers less than 150 between two whole numbers.	d) Estimate square or cube roots of numbers less than 1,000 between two whole numbers.	Other

<table>
<tr><th colspan="4">Num – 3. Number operations</th></tr>
<tr><th>Grade 4</th><th>Grade 8</th><th>Grade 12</th><th>Inherent Practice(s)</th></tr>
<tr><td>a) Add and subtract using conventional or unconventional procedures (e.g., strategic decomposing and composing):
• Whole numbers, or
• Fractions and mixed numbers with like denominators.</td><td>a) Perform computations with rational numbers.</td><td>a) Find integer or simple rational powers of real numbers.</td><td>Other</td></tr>
<tr><td>b) Multiply numbers using conventional or unconventional procedures (e.g., strategic decomposing and composing):
• Whole numbers no larger than two digits by two digits with paper and pencil computation, or
• Larger whole numbers using a calculator, or
• Multiplying a fraction by a whole number.</td><td></td><td>b) Perform arithmetic operations with real numbers, including common irrational numbers.</td><td>Other</td></tr>
<tr><td>c) Divide whole numbers:
• Up to three digits by one digit with paper and pencil computation, or
• Up to five digits by two digits with use of calculator.</td><td></td><td>c) Perform arithmetic operations with expressions involving absolute value.</td><td>Other</td></tr>
<tr><td></td><td>d) Describe the effect of operations on size, including the effect of attempts to multiply or divide a rational number by:
• Zero, or
• A number less than zero, or
• A number between zero and one, or
• One, or
• A number greater than one.</td><td>d) Describe the effect of multiplying and dividing by numbers including the effect of attempts to multiply or divide a real number by:
• Zero, or
• A number less than zero, or
• A number between zero and one, or
• One, or
• A number greater than one.</td><td>Abstracting and Generalizing</td></tr>
</table>

Num – 3. Number operations (continued)			
Grade 4	**Grade 8**	**Grade 12**	**Inherent Practice(s)**
e) Interpret, explain, or justify whole number operations and explain the relationships between them.	e) Interpret, explain, or justify rational number operations and explain the relationships between them.	e) *Analyze or interpret a proof by mathematical induction of a simple numerical relationship.	Justifying and Proving
f) Solve problems involving whole numbers and fractions with like denominators.	f) Solve problems involving rational numbers and operations using exact answers or estimates as appropriate.	# f) Solve problems involving numbers, including rationals and common irrationals.	Variable

Num – 4. Ratios and proportional reasoning			
Grade 4	**Grade 8**	**Grade 12**	**Inherent Practice(s)**
	a) Use ratios to describe problem situations.		Representing
	b) Use fractions to represent and express ratios and proportions.		Representing
	c) Use proportional reasoning to model and solve problems (including rates and scaling).	# c) Use proportions to solve problems (including rates of change and per capita problems).	Representing; Abstracting and Generalizing
	d) Solve problems involving percentages (including percent increase and decrease, interest rates, tax, discount, tips, or part/whole relationships).	# d) Solve multistep problems involving percentages, including compound percentages.	Variable

Num – 5. Properties of number and operations			
Grade 4	**Grade 8**	**Grade 12**	**Inherent Practice(s)**
a) Identify odd and even numbers.			Other
b) Identify factors of whole numbers.	b) Recognize, find, or use factors, multiples, or prime factorization.		Other
	c) Recognize or use prime and composite numbers to solve problems.	c) Solve problems using factors, multiples, or prime factorization.	Variable
	d) Use divisibility or remainders in problem settings.	# d) Use divisibility or remainders in problem settings.	Variable
e) Apply basic properties of operations.	e) Apply basic properties of operations, including conventions about the order of operations as applied to integers and rational numbers.	e) Apply basic properties of operations, including conventions about the order of operations as applied to real numbers.	Variable
		f) Recognize properties of the number system (whole numbers, integers, rational numbers, real numbers, and *complex numbers) and how they are related to each other and identify examples of each type of number.	Abstracting and Generalizing

Measurement

Meas – 1. Measuring physical attributes			
Grade 4	**Grade 8**	**Grade 12**	**Inherent Practice(s)**
a) Identify the attribute that is appropriate to measure in a given situation.			Other
b) Compare objects with respect to a given attribute, such as length, area, capacity, time, or temperature.	b) Compare objects with respect to length, area, volume, angle measurement, weight, or mass.	# b) Determine the effect of proportions and scaling on length, area, and volume.	Other
c) Estimate the size of an object with respect to a given measurement attribute (e.g., length, perimeter, or area using a grid).	c) Estimate the size of an object with respect to a given measurement attribute (e.g., area).	# c) Estimate or compare perimeters or areas of two-dimensional geometric figures.	Other
		d) Solve problems of angle measure, including those involving triangles or other polygons or parallel lines cut by a transversal.	Variable
e) Select or use appropriate measurement instruments such as ruler, meter stick, clock, thermometer, or other scaled instruments.	e) Select or use appropriate measurement instruments to determine or create a given length, area, volume, angle, weight, or mass.		Other
f) Solve problems involving perimeter of plane figures.	f) Solve mathematical or real-world problems involving perimeter or area of plane figures such as triangles, rectangles, circles, or composite figures.	f) Solve problems involving perimeter or area of plane figures such as polygons, circles, or composite figures.	Variable
g) Solve problems involving area of squares and rectangles.			Variable
	h) Solve problems involving volume or surface area of rectangular solids, and volume of right cylinders and prisms, or composite shapes	# h) Solve problems by determining, estimating, or comparing volumes or surface areas of three-dimensional figures.	Variable
	i) Solve problems involving rates and ratios such as speed or population density.	# i) Solve problems involving rates and ratios such as speed, density, population density, or flow rates.	Variable

Meas – 2. Systems of measurement			
Grade 4	**Grade 8**	**Grade 12**	**Inherent Practice(s)**
a) Select or use an appropriate type of unit for the attribute being measured such as length, angle size, time, or temperature.	a) Select or use an appropriate type of unit for the attribute being measured such as length, area, angle, time, or volume.	# a) Choose appropriate units for geometric measurements (length, area, perimeter, volume) and apply units in expressions, equations, and problem solutions.	Other
b) Solve problems involving conversions within the same measurement system such as conversions involving inches and feet or hours and minutes.	b) Solve problems involving conversions within the same measurement system such as conversions involving square inches and square feet.	# b) Solve problems involving conversions within or between measurement systems, given a relationship between the units.	Variable
	c) Estimate the measure of an object in one system given the measure of that object in another system and the approximate conversion factor. For example: • Distance: 1 kilometer is approximately 0.6 mile. • Money: U.S. dollars to Canadian dollars. • Temperature: Fahrenheit to Celsius.		Other
d) Determine appropriate unit of measurement in problem situations involving such attributes as length, time, capacity, or weight.	d) Determine appropriate unit of measurement in problem situations involving such attributes as length, area, or volume.	# d) Understand that numerical values associated with measurements of physical quantities are approximate, subject to variation, and must be assigned units of measurement.	Other
		# e) Determine appropriate accuracy of measurement in problem situations (e.g., the accuracy of measurement of the dimensions to obtain a specified accuracy of area) and find the measure to that degree of accuracy.	Variable
	f) Construct or solve problems (e.g., floor area of a room) involving scale drawings.	# f) Construct or solve problems involving scale drawings.	Variable

Meas – 3. Measurement in triangles			
Grade 4	**Grade 8**	**Grade 12**	**Inherent Practice(s)**
		# a) Solve problems involving indirect measurement.	Variable
		b) Solve problems using the fact that trigonometric ratios (sine, cosine, and tangent) stay constant in similar triangles.	Variable
		c) Use the definitions of sine, cosine, and tangent as ratios of sides in a right triangle to solve problems about length of sides and measure of angles.	Variable
		d) * Interpret and use the identity $\sin^2\theta + \cos^2\theta = 1$ for angles θ between 0° and 90°; recognize this identity as a special representation of the Pythagorean theorem.	Variable
		e) * Determine the radian measure of an angle and explain how radian measurement is related to a circle of radius 1.	Abstracting and Generalizing
		f) * Use trigonometric formulas such as addition and double angle formulas.	Variable
		g) * Use the law of cosines and the law of sines to find unknown sides and angles of a triangle.	Variable
		h) * Interpret the graphs of the sine, cosine, and tangent functions with respect to periodicity and values of these functions for multiples of $\pi/6$ and $\pi/4$.	Variable

Geometry

Geom – 1. Dimension and shape			
Grade 4	**Grade 8**	**Grade 12**	**Inherent Practice(s)**
a) Identify or describe (informally) real-world objects using simple plane figures (e.g., triangles, rectangles, squares, and circles) and simple solid figures (e.g., cubes, spheres, and cylinders).	a) Identify a geometric object given a written description of its properties.		Representing
b) Identify or draw angles and other geometric figures in the plane.	b) Identify, define, or describe geometric shapes in the plane and in three-dimensional space given a visual representation.	b) Give precise mathematical descriptions or definitions of geometric shapes in the plane and in three-dimensional space.	Other
	c) Draw or sketch from a written description polygons, circles, or semicircles.	c) Draw or sketch from a written description plane figures and planar images of three-dimensional figures.	Representing
		# d) Use two-dimensional representations of three-dimensional objects to visualize and solve problems.	Representing
e) Describe or distinguish among attributes of two- and three-dimensional shapes.	e) Demonstrate an understanding of two- and three-dimensional shapes in the world through identifying, drawing, reasoning from visual representations, composing, or decomposing.	# e) Analyze properties of three-dimensional figures including prisms, pyramids, cylinders, cones, spheres, and hemispheres.	Representing; Abstracting and Generalizing; Justifying and Proving

Geom – 2. Transformation of figures and preservation of properties			
Grade 4	**Grade 8**	**Grade 12**	**Inherent Practice(s)**
	a) Identify lines of symmetry in plane figures or recognize and classify types of symmetries of plane figures.	a) Recognize or identify types of symmetries (e.g., translation, reflection, rotation) of two- and three-dimensional figures.	Other
		b) Give or recognize the precise mathematical relationship (e.g., congruence, similarity, orientation) between a figure and its image under a transformation.	Other
	c) Recognize or informally describe the effect of a transformation (reflection, rotation, translation, or dilation) on two-dimensional figures.	c) Perform or describe the effect of a single transformation (reflection, rotation, translation, or dilation) on two- or three-dimensional geometric figures.	Other
d) Recognize attributes (such as shape and area) that do not change when plane figures are subdivided and rearranged.	d) Predict results of combining, subdividing, and recombining shapes of plane figures and solids (e.g., paper folding, tiling, subdividing and rearranging the pieces).	d) Identify transformations of shapes that preserve the area of two-dimensional figures or the volume of three-dimensional figures.	Justifying and Proving
	e) Justify relationships of congruence and similarity and apply these relationships using scaling and proportional reasoning.	e) Justify relationships of congruence and similarity and apply these relationships using scaling, proportional reasoning, and established theorems.	Justifying and Proving
	f) Apply the relationships among angle measures, lengths, and perimeters among similar figures.	f) Apply the relationships among angle measures, lengths, perimeters, and volumes among similar figures.	Variable
		g) Perform or describe the effects of successive (composites of) isometries and/or similarity transformations.	Representing; Justifying and Proving

Geom – 3. Relationships between geometric figures			
Grade 4	**Grade 8**	**Grade 12**	**Inherent Practice(s)**
a) Analyze or describe patterns in polygons when the number of sides increases, or the size or orientation changes.			Abstracting and Generalizing
b) Combine simple plane shapes to construct a given shape.	b) Apply geometric properties and relationships in solving problems in two and three dimensions.	b) Apply geometric properties and relationships to solve problems in two and three dimensions.	Justifying and Proving
c) Recognize two-dimensional faces of three-dimensional shapes.	c) Represent problem situations with geometric figures to solve problems.	# c) Represent problem situations with geometric figures to solve problems.	Representing; Abstracting and Generalizing
	d) Use the Pythagorean theorem to solve problems in two-dimensional situations.	# d) Use the Pythagorean theorem to solve problems in two- or three-dimensional situations.	Abstracting and Generalizing; Justifying and Proving
		e) Recall and interpret or use definitions and basic properties of congruent and similar triangles, quadrilaterals, and other polygons; circles; parallel, perpendicular and intersecting lines; and associated angle relationships (e.g., in solving problems or creating proofs).	Justifying and Proving; Mathematical Modeling
f) Describe and compare properties of simple and compound figures composed of triangles, squares, and rectangles.	f) Describe, compare, or analyze attributes of, or relationships between, triangles, quadrilaterals, and other polygonal plane figures.	f) Analyze attributes or relationships of triangles, quadrilaterals, and other polygonal plane figures.	Abstracting and Generalizing
	g) Describe or analyze properties and relationships of parallel or intersecting lines.	g) Analyze properties and relationships of parallel, perpendicular, or intersecting lines, including the angle relationships that arise in these cases.	Abstracting and Generalizing
		h) Make, test, and validate geometric conjectures using a variety of methods, including deductive reasoning and counterexamples.	Justifying and Proving

Geom – 3. Relationships between geometric figures (continued)			
Grade 4	**Grade 8**	**Grade 12**	**Inherent Practice(s)**
		i) * Analyze properties of circles and the intersections of lines and circles (inscribed angles, central angles, tangents, secants, and chords).	Abstracting and Generalizing; Justifying and Proving

Geom – 4. Position, direction, and coordinate geometry			
Grade 4	**Grade 8**	**Grade 12**	**Inherent Practice(s)**
a) Describe relative positions of points and lines using the geometric ideas of parallelism or perpendicularity.	a) Describe relative positions of points and lines using the geometric ideas of midpoint, points on a common line through a common point, parallelism, or perpendicularity.	a) Solve problems involving the coordinate plane using distance between two points, the midpoint of a segment, or slopes of perpendicular or parallel lines.	Variable
	b) Describe the intersection of two or more geometric figures in the plane (e.g., intersection of a circle and a line).	b) Describe the intersections of lines in the plane and in space, of a line and a plane, or of two planes in space.	Abstracting and Generalizing
	c) Visualize or describe the cross section of a solid.	c) Describe or identify conic sections and other cross sections of solids.	Abstracting and Generalizing
	d) Represent geometric figures using rectangular coordinates on a plane.	d) Represent two-dimensional figures algebraically using coordinates and/or equations.	Representing
		e) * Use vectors to represent velocity and direction; multiply a vector by a scalar and add vectors both algebraically and graphically.	Representing
		f) Find an equation of a circle given its center and radius and, given an equation of a circle, find its center and radius.	Other
		g) * Graph or determine equations for images of lines, circles, parabolas, and other curves under translations and reflections in the coordinate plane.	Representing; Justifying and Proving
		h) * Represent situations and solve problems involving polar coordinates.	Abstracting and Generalizing

Data Analysis, Statistics, and Probability

Data – 1. Data representation			
Grade 4	**Grade 8**	**Grade 12**	**Inherent Practice(s)**
Representations of data are indicated for each grade level in the next row. For some objectives, only a subset of the representations is applicable, indicated by a parenthetical list at the end of the objective.			
Pictographs, bar graphs, dot plots, tables, and tallies.	Histograms, plots over time, dot plots, scatterplots, box plots, bar graphs, circle graphs, stem and leaf plots, frequency distributions, and tables.	Histograms, plots over time, dot plots, scatterplots, box plots, bar graphs, circle graphs, stem and leaf plots, frequency distributions, and tables, including two-way tables.	
a) Read or interpret a single distribution of data.	a) Read or interpret data, including interpolating or extrapolating from data.	# a) Read or interpret graphical or tabular representations of data.	Representing; Abstracting and Generalizing
b) For a given distribution of data, complete a graph (limits of time make it difficult to construct graphs completely).	b) For a given distribution of data, complete a graph and solve a problem using the data in the graph (histograms, plots over time, dot plots, scatterplots, bar graphs, circle graphs).	# b) For a given set of data, complete a graph and solve a problem using the data in the graph (histograms, plots over time, dot plots, scatterplots).	Representing
c) Answer statistical questions by estimating and computing within a single distribution of data.	c) Answer statistical questions by estimating and computing with data from a single distribution or across distributions of data.	c) Answer statistical questions involving univariate or bivariate distributions of data.	Other
	d) Given a graphical or tabular representation of a distribution of data, determine whether the information is represented effectively and appropriately (histograms, plots over time, dot plots, scatterplots, box plots, bar graphs, circle graphs).	# d) Analyze, compare, and contrast different graphical representations of univariate and bivariate data (e.g., identify misleading uses of data in real-world settings and critique different ways of presenting and using information).	Representing; Justifying and Proving
		# e) * Organize and display data in a spreadsheet in order to recognize patterns and solve problems.	Representing

Data – 2. Characteristics of data sets			
Grade 4	**Grade 8**	**Grade 12**	**Inherent Practice(s)**
	a) Calculate, use, or interpret mean, median, mode, range, or shape of a distribution of data.	# a) Calculate, interpret, or use summary statistics for distributions of data including measures of center (mean, median), position (quartiles, percentiles), spread (range, interquartile range, variance, and standard deviation) or shape (skew, uniform, uni-/bimodal).	Representing
b) Given a distribution of whole number data in a context, identify and explain the meaning of the greatest value, of the least value, or of any clustering or grouping of data in the distribution.	b) Describe a distribution of data using its mean, median, mode, range, interquartile range, and shape.	b) Recognize how linear transformations of one-variable data affect mean, median, mode, range, interquartile range, and standard deviation.	Abstracting and Generalizing
	c) Identify outliers and determine their effect on the mean, median, mode, or range.	# c) Determine the effect of outliers on the mean, median, mode, range, interquartile range, or standard deviation.	Abstracting and Generalizing
	d) Using appropriate statistical measures, compare two or more data sets describing the same characteristic for two different populations or subsets of the same population.	d) Compare data sets using summary statistics (mean, median, mode, range, interquartile range, shape, or standard deviation) describing the same characteristic for two different populations or subsets of the same population.	Other
	e) Visually choose the line that best fits given a scatterplot and informally explain the meaning of the line. Use the line to make predictions.	e) Approximate a trend line if a linear pattern is apparent in a scatterplot or use a graphing calculator to determine a least-squares regression line and use the line or equation to make predictions.	Representing; Justifying and Proving

Data – 2. Characteristics of data sets (continued)			
Grade 4	**Grade 8**	**Grade 12**	**Inherent Practice(s)**
		# f) Recognize or explain how an argument based on data might confuse correlation with causation.	Justifying and Proving
		g) * Identify and interpret the key characteristics of a normal distribution such as shape, center (mean), and spread (standard deviation).	Abstracting and Generalizing
		# h) * Recognize and explain the potential errors that can arise when extrapolating from data.	Justifying and Proving

Data – 3. Experiments and samples			
Grade 4	**Grade 8**	**Grade 12**	**Inherent Practice(s)**
	a) Given a sample, identify possible sources of bias in sampling.	# a) Identify possible sources of bias in sample survey populations or questions and describe how such bias can be controlled and reduced.	Mathematical Modeling
	b) Distinguish between a random sample and a nonrandom sample.	b) Recognize and describe a method to select a simple random sample.	Mathematical Modeling
		# c) Draw inferences from samples, such as estimates of proportions in a population, estimates of population means, or decisions about differences in means for two "treatments."	Abstracting and Generalizing
		d) Identify or evaluate the characteristics of a good survey or of a well-designed experiment.	Mathematical Modeling
		e) * Recognize the differences in design and in conclusions between randomized experiments and observational studies.	Justifying and Proving

Data – 4. Probability			
Grade 4	**Grade 8**	**Grade 12**	**Inherent Practice(s)**
		# a) Determine whether two events are independent or dependent.	Other
	b) Using assumption of randomness, determine the theoretical probability of simple or compound events in familiar contexts.	# b) Using assumptions such as randomness, determine the theoretical probability of simple or compound events in familiar or unfamiliar contexts.	Other
	c) Given the results of an experiment or simulation, estimate the probability of simple and compound events in familiar contexts.	# c) Given the results of an experiment or simulation, estimate the probability of simple or compound events in familiar or unfamiliar contexts.	Other
	d) Use theoretical probability to evaluate or predict experimental outcomes in familiar contexts.	# d) Use theoretical probability to evaluate or predict experimental outcomes in familiar or unfamiliar contexts.	Justifying and Proving; Mathematical Modeling
	e) Determine the sample space for a given situation.	e) Determine the number of ways an event can occur using tree diagrams, formulas for combinations and permutations, or other counting techniques.	Representing
	f) Use a sample space to determine the probability of possible outcomes for an event.		Other

Data – 4. Probability (continued)			
Grade 4	**Grade 8**	**Grade 12**	**Inherent Practice(s)**
	g) Represent the probability of a given outcome using fractions, decimals, and percents.		Other
	h) Determine the probability of independent and dependent events. (Dependent events should be limited to a small sample size.)	h) Determine the probability of independent and dependent events.	Other
		i) Determine conditional probability using two-way tables.	Representing
	j) Interpret and apply probability concepts to practical situations, and simple games of chance.	# j) Interpret and apply probability concepts to practical situations, including odds of success or failure in simple lotteries or games of chance.	Representing
		k) * Use the binomial theorem to solve problems.	Variable

Algebra

Alg – 1. Patterns, relations, and functions			
Grade 4	**Grade 8**	**Grade 12**	**Inherent Practice(s)**
a) Recognize, describe (in words or symbols), or extend simple numerical and visual patterns.	a) Recognize, describe, or extend numerical and visual patterns using tables, graphs, words, or symbols.	a) Recognize, describe, or extend numerical patterns, including arithmetic and geometric sequences (progressions).	Abstracting and Generalizing
		b) Express linear and exponential functions in recursive and explicit form given a verbal description, table, or some terms of a sequence.	Abstracting and Generalizing
c) Given a description, extend or find a missing term in a pattern or sequence.	c) Examine or create patterns, sequences, or linear functions expressed as a rule numerically, verbally, or symbolically.		Abstracting and Generalizing
d) Create a different representation of a pattern or sequence given a verbal description.			Representing
	e) Identify functions as linear or nonlinear or contrast distinguishing properties of functions from tables, graphs, or equations.	e) Identify or analyze distinguishing properties of linear, quadratic, rational, exponential, or *trigonometric functions from tables, graphs, or equations.	Other
	f) Interpret the meaning of slope or intercepts, or determine the rate of change between two points on a graph of a linear function.		Other
		g) Determine whether a relation, given in verbal, symbolic, tabular, or graphical form, is a function.	Representing
		h) Recognize and analyze the general forms of linear, quadratic, rational, exponential, or *trigonometric functions.	Representing
		i) Determine the domain and range of functions given in various forms and contexts.	Other

Alg – 1. Patterns, relations, and functions (continued)			
Grade 4	**Grade 8**	**Grade 12**	**Inherent Practice(s)**
		j) * Given a function, determine its inverse if it exists and explain the contextual meaning of the inverse for a given situation.	Abstracting and Generalizing

Alg – 2. Algebraic representations			
Grade 4	**Grade 8**	**Grade 12**	**Inherent Practice(s)**
a) Translate between different representational forms (symbolic, numerical, verbal, or pictorial) of whole number relationships (such as from a written description to an equation or from a function table to a written description).	a) Translate between different representations of linear expressions using symbols, graphs, tables, diagrams, or written descriptions.	a) Create and translate between different representations of algebraic expressions, equations, and inequalities (e.g., linear, quadratic, exponential, or *trigonometric) using symbols, graphs, tables, diagrams, or written descriptions.	Representing
	b) Interpret and compare representations of linear relationships expressed in symbols, graphs, tables, diagrams, or written descriptions.	# b) Interpret and compare representations of relationships expressed in symbols, graphs, tables, diagrams (including Venn diagrams), or written descriptions.	Representing
	c) Graph or interpret points represented by ordered pairs of numbers on a rectangular coordinate system.		Representing
	d) Solve problems involving coordinate pairs on the rectangular coordinate system.	d) Perform or interpret transformations on the graphs of linear, quadratic, exponential, and *trigonometric functions.	Representing

Alg – 2. Algebraic representations (continued)			
Grade 4	**Grade 8**	**Grade 12**	**Inherent Practice(s)**
		e) Make inferences or predictions using an algebraic model of a situation.	Abstracting and Generalizing; Justifying and Proving; Mathematical Modeling
	f) Identify or represent functional relationships in meaningful contexts including proportional, linear, and common nonlinear relationships (e.g., compound interest, bacterial growth) in tables, graphs, words, or symbols.	# f) Given a real-world situation, determine if a linear, quadratic, rational, exponential, *logarithmic, or *trigonometric function fits the situation.	Representing
		# g) Solve problems involving exponential growth and decay.	Variable
		h) *Identify distinguishing characteristics of exponential, logarithmic, and rational functions (e.g., discontinuity, asymptotes, concavity).	Other

Alg – 3. Variables, expressions, and operations			
Grade 4	**Grade 8**	**Grade 12**	**Inherent Practice(s)**
a) Use letters and symbols to represent an unknown quantity in a simple mathematical expression.			Representing
b) Express simple mathematical relationships using expressions, equations, or inequalities.	b) Write algebraic expressions, equations, or inequalities to represent a situation.	b) Write algebraic expressions, equations, or inequalities to represent a situation.	Representing
	c) Perform basic operations, using appropriate tools, on linear algebraic expressions (including grouping and order of multiple operations involving basic operations, exponents, roots, simplifying, and expanding).	c) Perform basic operations, using appropriate tools, on algebraic expressions including polynomial and rational expressions.	Other
		d) Write equivalent forms of algebraic expressions, equations, or inequalities to represent and explain mathematical relationships.	Representing; Abstracting and Generalizing
		# e) Evaluate algebraic expressions, including polynomials and rational expressions.	Other
		f) Use function notation to evaluate a function at a specified point in its domain and combine functions by addition, subtraction, multiplication, division, and composition.	Other
		g) * Determine the sum of finite and infinite arithmetic and geometric series.	Abstracting and Generalizing
		h) Use basic properties of exponents and *logarithms to solve problems.	Other

Alg – 4. Equations and inequalities			
Grade 4	**Grade 8**	**Grade 12**	**Inherent Practice(s)**
a) Find the unknown(s) in a whole number sentence (e.g., in an equation or simple inequality like [_] + 3 > 7).	a) Solve linear equations or inequalities (e.g., Solve for x in $ax + b = c$ or $ax + b = cx + d$ or $ax + b > c$).	a) Solve linear, rational, or quadratic equations or inequalities, including those involving absolute value.	Other
b) Interpret "=" as an equivalence between two values and use this interpretation to solve problems.		b) * Determine the role of hypotheses, logical implications, and conclusions in algebraic arguments about equality and inequality.	Other
c) Verify a conclusion using simple algebraic properties derived from work with numbers (e.g., commutativity, properties of 0 and 1).	c) Make, validate, and justify conclusions and generalizations about linear relationships.	c) Use algebraic properties to develop a valid mathematical argument.	Justifying and Proving
	d) Analyze situations or solve problems using linear equations and inequalities with rational coefficients symbolically or graphically (e.g., $ax + b = c$ or $ax + b = cx + d$).	# d) Analyze situations, develop mathematical models, or solve problems using linear, quadratic, exponential, or *logarithmic equations or inequalities symbolically or graphically.	Representing; Mathematical Modeling
	e) Interpret relationships between symbolic linear expressions and graphs of lines by identifying and computing slope and intercepts (e.g., in $y = ax + b$, know that a is the rate of change and b is the vertical intercept).	e) Solve (symbolically or graphically) a system of equations or inequalities and recognize the relationship between the analytical solution and graphical solution.	Representing
	f) Use and evaluate common formulas (e.g., relationship between a circle's circumference and diameter, $C = \pi d$, distance and time under constant speed).	# f) Solve problems involving special formulas such as: $A = P(I + r)^t$ or $A = Pe^{rt}$.	Mathematical Modeling
		# g) Solve an equation or formula involving several variables for one variable in terms of the others.	Other
		h) * Solve quadratic equations with complex roots.	Other

Three special studies are proposed to support the 2026 NAEP Mathematics Framework. Special studies play a unique and essential role in the NAEP Mathematics ecosystem: advancing the vision set forth in the Framework. Some components of that vision may be aspirational – policies or processes that are essential for valid and reliable assessment of mathematics knowledge and skills, but that require advancements in assessment design, research, or technology to support implementation at scale.

Advancing the Assessment of NAEP Mathematical Practices and Mathematical Literacy

As a group, the NAEP Mathematical Practices introduced in Chapter 3 constitute an ideal topic area for special study. The practices are new to the Framework and a defining feature of the vision it presents. To ensure that this vision is executed fully, NAEP should advance several strands of research and development, which correspond with the studies described in this appendix. First, NAEP should assess mathematical practices in authentic settings that resemble real-world problems where the practices are often applied. This strand is addressed in Study 1. Second, NAEP should signal the value of mathematical practices in the same way most large-scale assessment programs promote the importance of the skills they measure – by reporting results. This strand is addressed in Study 2. Third, the Framework emphasizes mathematical literacy as an essential component of mathematics knowledge and skills that NAEP Mathematics items can and should target. Mathematical literacy is not, however, included formally as one of the five content areas or as one of the five NAEP Mathematical Practices. Therefore, a third strand of research should focus on the extent to which mathematical literacy can be measured and reported – accurately and reliably – under the requirements and constraints of a NAEP Mathematics operational administration. This strand is addressed in Study 3.

Study 1: Assessing Mathematical Practice in Context

Overview

Study 1 will examine ways to measure the NAEP Mathematical Practices by leveraging the rich data that scenario-based tasks and discrete items can generate in a digital assessment environment. The first phase of the study will establish a baseline by examining measurement features (e.g., content coverage, discrimination) of scenario-based tasks and other context-situated items linked to each NAEP Mathematical Practice. The second phase of the study will collect process data (e.g., activity logs), which will be recorded as students interact with elements of each situation (i.e., scenarios and discrete item contexts). This phase will also explore research-based methods for using process data to generate measures of student performance. Ultimately, this study will help NAEP not only determine the feasibility of capturing new process data through existing item types, but also gauge the measurement value of process data, either compared against or combined with response data.

Rationale

As interest in mathematical practices has grown, so too has the need for assessment approaches that allow students to demonstrate mastery of those practices. Under traditional item-delivery models, the steps that take place between the presentation of the item and the student's response are invisible in the assessment process. If those interim steps are meaningful to the assessed construct – in this case, a NAEP Mathematical Practice – then decomposing items into their constituent steps could help sharpen inferences about a practice.

A new family of mathematics assessment approaches has emerged in response to this need, its defining feature being multiple points of measurement (e.g., item collections), often connected to a common stimulus or problem, and in some cases building on previous steps. These multistep approaches are particularly well suited to the assessment of mathematical practices, because success depends not on the recollection of an isolated fact or theorem, but rather on the skill to draw upon multiple mathematics domains and solve complicated problems requiring multiple steps. These multistep approaches are also particularly well suited to NAEP Mathematics, since the Framework places the NAEP Mathematical Practices alongside mathematics content as fundamental elements of mathematics assessment. This special study is intended to advance that vision, not only by leveraging current techniques for assessing practices, but also by extending those assessment techniques to learn more about students' response processes.

For NAEP Mathematics, the multistep approach will be accomplished through scenario-based tasks as well as through other context-situated digitally based items. Study 1 will use these items as a starting point and explore extensions that could generate even richer student performance data. For example, the typical scenario-based task on a NAEP Science or NAEP TEL assessment is a cluster of items that ask students to complete a series of steps related to the same underlying scenario. Like any other collection of items, a scenario-based task yields a group of item scores for each student, allowing NAEP to present students with engaging multistep problems while maintaining the same fundamental approaches to item scoring and psychometric scaling that are applied to other items.

Scenario-based tasks offer a promising avenue for measuring the NAEP Mathematics Practices, so the first phase of Study 1 will involve examining the measurement information that scenario-based tasks produce across content areas and across the performance continuum. More importantly, the first phase will provide a baseline, characterizing the information that NAEP items and tasks provide about mathematical practices through *response* data (i.e., students' scored responses to a discrete item or to the standardized group of items administered through a scenario-based task). If students' navigations through and interactions with these items' contexts are summarized only through response data, available measurement information may be under-utilized. Therefore, the second phase of the study will explore what additional information can be gained from *process* data.

In brief, process data are recordings of students' interactions with a digital environment. Clickstream data, activity logs, text, and transcribed voice responses are all examples. Once recorded, process data can be analyzed using a variety of statistical methods to produce measures of mathematical practice according to a standardized set of rules. One potential advantage to collecting and analyzing process data is the insight these data provide about students' actions that are not part of a formal item but that are nonetheless relevant to mathematical practice. In addition, process data are collected passively, recording students' interactions rather than pausing a scenario to deliver an item, which could improve time efficiency.

However, process data also present new complexities. Student privacy concerns and available technology could each limit the variety and usefulness of available process data. In addition, process data are by definition untethered to item-writing rules and content targets, so establishing evidence of content validity (e.g., the alignment of a finite set of items to the NAEP Mathematics ALDs) for process data may prove challenging.

Nonetheless, it is reasonable to expect that an initial collection of analysis rules for process data could be developed as a component of Study 1, with each rule specifying what evidence a given analytical procedure (e.g., natural language processing) is designed to elicit and how that evidence maps to the 2026 NAEP Mathematics ALDs, which now include explicit references to the five NAEP Mathematical Practices. These analysis rules would function the same way item-development and scoring-rubric guidelines do for conventional NAEP Mathematics items, providing a clear link between the ALDs and students' response processes (for relatively structured assessment tasks, such as responding to a multiple choice item, or for relatively unstructured tasks, such as interacting with a digital environment). In fact, process data may increase the measurement information that can be gleaned from discrete items and scenario-based tasks, by generating even richer data at a smaller grain size.

Outcomes

Study 1 will produce three key outcomes in service of the 2026 Mathematics Framework's vision for the NAEP Mathematical Practices:

1. NAEP will characterize the measurement properties (e.g., content coverage, discrimination, potential bias, assessment time relative to measurement information) of items and tasks *as they relate to the NAEP Mathematical Practices*. NAEP currently collects these data for mathematics content areas, but the 2026 Framework is the first to explicitly include attention to five NAEP Mathematical Practices.
2. NAEP will determine the feasibility of collecting process data through different item types. A wide variety of process data have been suggested in the literature (e.g., Andrews et al., 2017), but the variety of process data that can be collected within the constraints of a NAEP administration may be more limited.
3. NAEP will compile preliminary information about the value of process data in comparison to and as a companion to conventional response data, in terms of relevant information about student performance in mathematical practices.

Study 2: Reporting Results for Mathematical Practices

Overview

Study 2 will examine ways to provide information about the NAEP Mathematical Practices to the general public. The first phase will involve researching commonly used approaches for communicating assessment results, conceptualizing a limited set of reporting options, and producing sample reports. The second phase will involve gathering feedback on reporting approaches through focus groups with stakeholders and, if practicable, conducting structured

A/B testing. This study is intended to produce feasible ways to provide information about the NAEP Mathematical Practices, under a key constraint: Unlike the content areas, the five NAEP Mathematical Practices will not be scaled independently. Therefore, given the absence of scale scores for the practices, NAEP should avoid formats for reporting that risk confusion or misinterpretation. Although reporting options for NAEP Mathematical Practices affect any decision to scale practices separately, this study will not address the feasibility of scaling for the NAEP Mathematics Practices.

Rationale

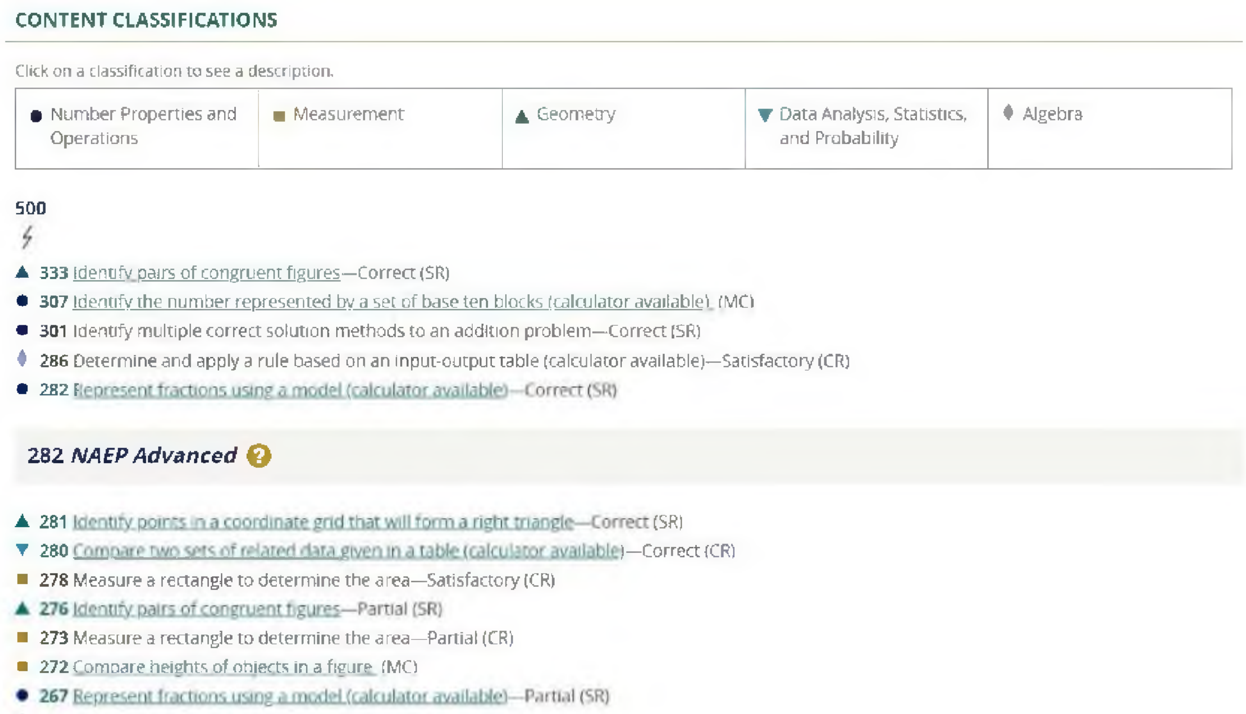
The 2026 NAEP Mathematics Framework defines five NAEP Mathematical Practices, articulates how those practices should be assessed in various content areas, and positions the practices as a core component of assessing student achievement in mathematics – critical information for educators, parents, policymakers, and assessment developers. An important next step, as emphasized in the Framework, will be sharing the results with the general public. Releasing information about NAEP items and the NAEP Mathematical Practices should underscore the practices' fundamental importance in NAEP assessment.

Since NAEP Mathematical Practices are intertwined with NAEP mathematics content areas, the 2026 Framework's Technical Advisory Committee recommended against creating separate reporting scales for each practice. Instead, student performance on items that assess NAEP Mathematical Practices may be communicated descriptively, drawing upon common reporting approaches in large-scale assessment programs. The first phase of Study 2 will involve compiling a list of candidate reporting approaches based on a scan of what is done for other large-scale assessments (including, of course, other NAEP assessment programs). One option for descriptive reporting (item maps) is described next. Note that this example is provided for illustrative purposes only, and not as a suggested reporting tool for the NAEP Mathematical Practices. Any reporting approach would need to be evaluated in terms of its cost, its appeal to stakeholders, and the extent to which it maximizes effective communication and minimizes misinterpretation.

Item Maps Example

NAEP uses item maps to help illustrate what students know and can do in a variety of subject areas, including mathematics. In an item map, items are placed along the NAEP scale in each grade level. An item's position depends on its difficulty, which is estimated empirically using student response data. Items associated with higher scale scores are more difficult, requiring higher levels of knowledge and skills for a correct response. An example item map for the 2017 NAEP Mathematics Assessment is presented in Illustration E.1. Each item's description focuses on the knowledge and skills needed to respond successfully, and "content classifications" icons refer to the specific content area being assessed. The same approach could be used to illustrate the relative difficulty of specific practices (by adding five NAEP Mathematical Practice icons).

Illustration E.1. NAEP Mathematics Grade 4 Item Map

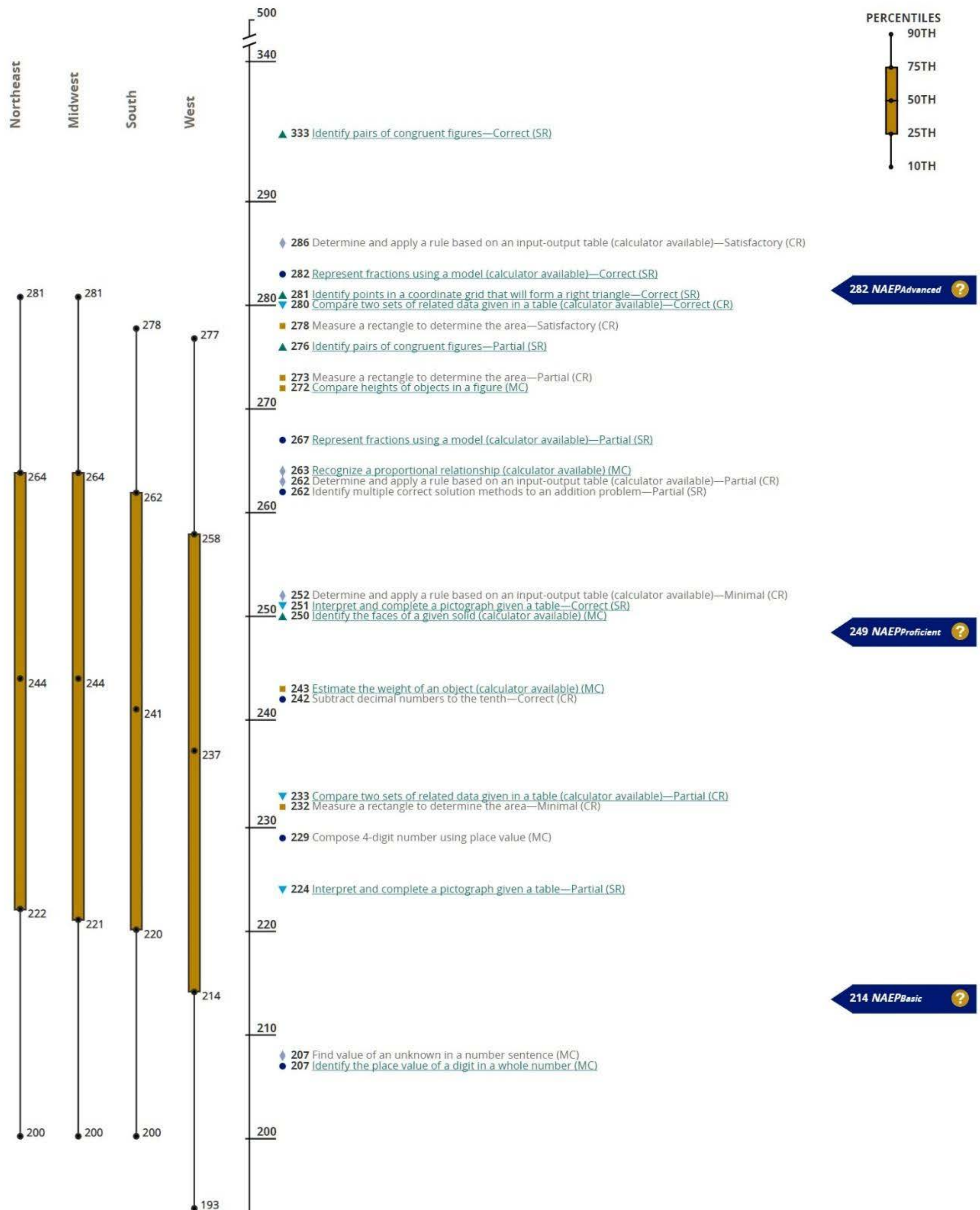


Source: https://www.nationsreportcard.gov/itemmaps/?subj=MAT&grade=4&year=2017

Item maps can be augmented to summarize student performance or to enable comparisons across student groups. In Illustration E.2, four box and whisker plots have been added to summarize student performance for the 2017 NAEP Mathematics Assessment in four U.S. regions. Another key component of the first phase of Study 2 would be estimating the time investment involved in each potential reporting solution. For example, in Illustrations E.1 and E.2, the addition of NAEP Mathematical Practices information would require new and existing NAEP items to be tagged with the practice(s) they feature, but would not require additional scaling or standard-setting procedures.

Adding NAEP Mathematical Practices to the figures in Illustrations E.1 and E.2 would create a somewhat crowded visual, with not only five content-area icons but also five NAEP Mathematical Practices icons. One alternative could be to create a separate item map for each content area, and then include items within that content area tagged with each practice. To simplify the presentation even further, items could be removed from the maps and NAEP Mathematical Practices could instead be summarized in box and whisker plots. For example, in Illustration E.2, NAEP Mathematical Practices could take the place of U.S. regions, and box and whisker plots would summarize the distribution of item difficulties associated with each practice. Again, this would require NAEP items to be tagged with practices, but would not require new scaling or standard setting.

Illustration E.2. NAEP Mathematics Grade 4 Item Map by U.S. Region



Source: https://www.nationsreportcard.gov/itemmaps/?subj=MAT&grade=4&year=2017&jurisdiction=NT&variable=CENSREG.

After a limited set of sample reports is created to represent the candidate reporting solutions, the second phase of Study 2 will involve report field-testing. A plausible first step in field-testing would be to convene geographically diverse stakeholder focus groups to solicit feedback on each report's clarity and simplicity, and on any areas that raise the risk of confusion or unintended interpretation. Focus group panelists may also be asked to provide their interpretations of the report data (anonymously, to avoid peer influence). Unexpected interpretations in the anonymous feedback may highlight problem areas requiring further evaluation and development. If practicable, A/B testing could be added in the second phase, offering a more formal approach to comparing interpretations. Typical A/B testing involves randomly assigning consumers of visual data to one of two (or more) formats; the accuracy of consumers' interpretations across the two groups can be compared to help identify the format that minimizes confusion and misinterpretation.

Outcomes

Study 2 will produce two key outcomes in service of the 2026 Mathematics Framework vision for mathematical practices:

1. NAEP will research common reporting approaches and then assess the viability of replicating or adapting those approaches in the NAEP context. This feasibility study will generate or supplement a set of practical considerations that NAEP can use when considering the adoption of other large-scale programs' reporting methods. In addition, this study will produce one or more candidate reporting methods for consideration by other NAEP programs seeking to report on domains without scale scores.
2. NAEP will determine useful and appropriate reporting formats for the NAEP Mathematical Practices. This will allow NAEP to signal the value of mathematical practices (an essential element of the Framework vision) without disseminating reports that risk widespread misinterpretation.

Study 3: Investigating Options for Assessing and Reporting Mathematical Literacy

Overview

Study 3 will focus on the extent to which mathematical literacy can be assessed and potentially reported via collections of NAEP Mathematics items and content objectives in grade 12. The first phase of the study will focus on the mathematical literacy construct itself; empirical analyses will help NAEP determine the precision and accuracy with which mathematical literacy can be measured and whether student performance in mathematical literacy constitutes a new dimension separate from the existing content areas and practices. Provided that students' mathematical literacy skills are separable from other content knowledge and practices, the second phase of the study will investigate options for reporting on mathematical literacy. The second phase, therefore, may share many design features and decision points with Study 2 (reporting on mathematical practices). Ultimately, this study will help NAEP determine the feasibility of assessing mathematical literacy and identify potential item-development or psychometric issues that would need to be addressed in order to do so.

Rationale

Relative to previous NAEP Mathematics Frameworks, the 2026 Framework increases the focus on the assessment of mathematical literacy, particularly in grade 12. First, the Framework provides a definition of mathematical literacy:

> Mathematical literacy is the application of numerical, spatial, or symbolic mathematical information to situations in a person's life as a community member, citizen, worker, or consumer.

As noted in the 2026 Framework, a variety of NAEP items assess student actions and knowledge that could be viewed as requiring mathematical literacy (e.g., making decisions about personal finances; understanding quantitative information in print and visual media; making the accurate measurements in order to prepare a meal). Mathematical literacy can be found in the objectives in grades 4 and 8, but, until the 2026 Framework, mathematical literacy at grade 12 had received comparatively little attention. In the 2026 Framework, some grade 12 objectives are identified with a number/hashtag sign (#), if there are everyday applications of the objective to situations in a person's life as a community member, citizen, worker, or consumer. These identifications have been included in the Framework for two reasons – to encourage the development of items measuring mathematical literacy and to support the identification of existing items in order to explore the feasibility of assessing and reporting on mathematical literacy.

This special study, therefore, is intended as a first step in the investigation of mathematical literacy as an assessable and reportable construct under the requirements and constraints of a NAEP Mathematics operational administration. Depending on the results of this study, future frameworks might identify mathematical literacy as a new content area at one or more grade levels. Alternatively, future frameworks may call for additional research, such as a special study focused on curriculum or assessment frameworks around the world that include mathematical literacy as a significant area.

Because this special study includes an analysis of options for reporting on mathematical literacy, it may share some common elements with Study 2 (reporting on mathematical practices). However, prior to considering reporting options, NAEP must first examine the assessability of mathematical literacy as a construct. Although it has been defined in the mathematics education literature and measured by other large-scale assessment programs, mathematical literacy is not currently a NAEP Mathematics content area or a NAEP Mathematical Practice. Therefore, the first phase of this study will focus on whether student performance in mathematical literacy is meaningfully different from performance in existing content areas and practices.

The educational measurement literature offers numerous well-understood and widely used methodologies (e.g., confirmatory factor analysis, Item Response Theory model-fit tests) for examining an assessment's dimensionality and the separability of the constructs it is meant to quantify. In addition to dimensionality tests, internal consistency statistics will provide lower-bound estimates of the reliability of students' mathematical literacy scores. This special study could also incorporate the judgment of subject-matter experts early in the assessment process by asking an independent group of mathematics content and assessment experts to identify mathematical literacy items among a larger set of items targeting various mathematics content areas and practices. If experts consistently distinguish mathematical literacy items from items that target other constructs, that would be promising (albeit incomplete) evidence of the degree to which mathematical literacy can be assessed as a unique aspect of mathematics knowledge and skill. This judgment-based study could also inform item development, highlighting the items and item features that promote identifications with mathematical literacy.

Depending on the results of the first phase of this special study, NAEP may next conduct a systematic investigation of options for *reporting* on mathematical literacy. The steps in this phase could largely mirror the design and sequence of Study 2. NAEP would first conduct a landscape scan, reviewing existing large-scale assessment programs' approaches to reporting on constructs with multiple subdomains. For example, the exploratory approach illustrated by item maps in Figures 2 and 3 may be suitable for mathematical literacy. If mathematical literacy is considered as a potential content area (rather than a practice), one reporting option would be to add a mathematical literacy icon to current NAEP Mathematics item maps. Alternatively, box and whisker plots could be presented to compare the distributions of mathematical literacy items that demand different NAEP Mathematical Practices (e.g., Representing versus Abstracting and Generalizing versus Mathematical Modeling). Then, similar to Study 2, focus groups and A/B testing can be employed to verify that the *intended* interpretations of a reporting format align with *actual* interpretations by consumers of the report.

Regardless of the specific methodology, it is important to emphasize the exploratory nature of Study 3. It is not intended to produce procedures for scaling mathematical literacy separately from existing content areas. Even if the findings from dimensionality and reliability analyses in the first phase suggest that it would be feasible, adding a subscale to the NAEP Mathematics Assessment would require extensive empirical analysis and deliberation. Rather, Study 3 represents a first step, providing foundational information about the role of mathematical literacy in the NAEP Mathematics Assessment.

Outcomes

This study will produce two key outcomes in service of the 2026 Mathematics Framework's vision for mathematical literacy:

1. NAEP will determine whether mathematical literacy is a unique dimension that can be measured accurately, reliably, and separately from each mathematics content area and each mathematical practice.
2. To signal the importance of mathematical literacy, NAEP will develop considerations for valid and straightforward reporting of mathematical literacy results.

Appendix F: Panel Process for NAEP Contextual Variables Analysis

The 2026 NAEP Mathematics Framework Development Panel was charged with reviewing and making recommendations for changes to student, teacher, and school administrator surveys designed to measure the NAEP mathematics-specific contextual variables. Those recommendations appear in Chapter 5 of the 2026 NAEP Mathematics Framework. The goal of the review and recommendations was increasing the usefulness of these data for the interpretation of NAEP Mathematics results in 2026 and beyond. In making these recommendations, the Development Panel adhered to guidance from the 2026 NAEP Mathematics Framework Visioning Panel:

> A major goal of NAEP [reporting] should involve understanding NAEP results through analyses of contextual variables, including opportunities to learn…to emphasize the fact that what students know and can do is profoundly shaped by the social, human, and material resources available for their learning, the contexts in which they live and learn, as well as teacher and student orientations, motivations, and beliefs.

Overview

Item development for the 2017 NAEP Mathematics Survey Questionnaires addressed two areas of context: opportunity to learn and non-cognitive student factors. The broad categories of contextual factors within these areas were arranged around four issues: *organization of instruction*, including curriculum content, instructional strategies, use of technology in instruction, and use of formative assessment; *resources for learning and instruction*, including people resources, product resources, and time resources; *student factors*, including mathematics activities outside of school, self-related beliefs, interest and motivation, grit for mathematics, and desire for learning for mathematics; and *teacher preparation*, including content knowledge and subject-specific training, professional development, and non-cognitive teacher factors.

Importantly, attention to each of these issues and associated subcategories was justified on the basis of its impact on student performance *specifically* in mathematics, its connection to possible interventions in and outside the classroom, and the extent to which information about it could appropriately be captured through survey questionnaires. For the 2026 NAEP Mathematics Framework update, the Development Panel was guided by the Visioning Panel's recommendation to use recent mathematics education literature on opportunity to learn and to explore the contextual variables associated with differing levels of performance among and within subgroups.

The Development Panel's subsequent review process entailed examining specific questions in the current student, teacher, and administrator surveys in each category and subcategory; analyzing these questions with respect to the extent to which they reflected recommendations from the Visioning Panel and implications of the current research on opportunities to learn; and then articulating recommendations for revising the focus of the NAEP mathematics–specific surveys in order to strengthen what could be learned from the contextual variables data.

Rationales for New Survey Emphases

In reviewing the current surveys, the Development Panel addressed not only the mathematics content students are taught, but also the opportunities students have to engage with that content in meaningful ways (NCTM, 2014). Do students engage in authentic mathematics activity focused on making sense of rich non-routine tasks? Do they have access to the kinds of tools that support the exploration of mathematical representations and how these representations connect to important mathematical ideas? Do they engage in meaningful discourse about their mathematical thinking and the mathematical thinking of others? In other words, to what extent do students have opportunities to engage with important *mathematics content and practices*, particularly the mathematical practices that are now recommended to be part of the NAEP framework? All of this colors how authentically students engage with opportunities to learn (e.g., Sword, Matsuura, Cuoco, Kang, & Gates, 2018). For these reasons, the Development Panel suggested addressing the extent to which students have opportunities to engage in authentic mathematical activities that provide opportunities to think and reason like mathematicians, both during mathematics instruction and with regard to the nature of assigned homework (e.g., Rosario et al., 2015).

The Development Panel's review was informed by the importance of determining students' relationship to schooling and mathematics (e.g., Boaler, 2010; Strutchens, 2000). How do students think about the strengths they bring to their mathematical work, including the extent to which they have developed a strong mathematical identity and sense of agency? Do all students feel welcome in their mathematics classroom? Do their ideas matter to the teacher and their classmates? Are they seen as capable mathematical thinkers who have contributions to make during group work and whole-class discussions? The answers to these questions shape how individual students see themselves as mathematical thinkers, inform the kind of mathematical identities individual students develop, and impact how deeply each student engages in the opportunities to learn that arise during classroom instruction (Steele, Spencer, & Aronson, 2002). For these reasons, the Development Panel suggested addressing *student mathematical identity* through questions addressing student participation in activities (such as discussion of mathematical ideas or evaluation of how a mathematics problem is framed), how students see themselves as mathematics learners, what they think it means to do mathematics, and what they think it takes to be a successful mathematics student.

In reviewing the literature and existing contextual issues, the Development Panel explored how to capture the extent to which students are given opportunities to draw on knowledge and skills acquired through mathematical experiences *outside* of the mathematics classroom as they engage with and make sense of activities *inside* the mathematics classroom (Aguirre, Mayfield-Ingram, & Martin, 2013; Boaler, 2002; Civil, 2007; Langer-Osuna, Moschkovich, Noren, Powell, & Vazquez, 2016; Lewis, 2014; Martin, 2006; Tomaz & David, 2015). Such funds of knowledge, acquired outside and used inside the classroom, have been defined as the skills and knowledge that have been historically and culturally developed to enable an individual or a household to function within a given culture. This includes a wide range of ethnic and language communities, and, importantly, such communities as street vendors, artists, video gamers, and even the unhoused, who have established ways of interacting mathematically with their environments. Recently, research on funds of knowledge has come to include individuals with disabilities who have created their own ways of successfully mathematizing their environments (Tan & Kastberg, 2017). To what extent are these wide-ranging experiences leveraged to strengthen student opportunity to learn in the mathematics classroom? The panel suggested addressing questions about mathematical activity outside of the mathematics classroom, connections between what students are learning during mathematics instruction and students' experiential worlds outside of school, and *instructional organization and strategies* related to these (Crespo, Celedon-Pattichis, & Civil, 2017; Fernandez, Crespo, & Civil, 2017; White, Fernandez, & Civil, 2017).

The Visioning Panel recommended that teacher questions on the contextual surveys parallel many of the questions on the student surveys, in order to be able to explore consistencies or inconsistencies across student and teacher perspectives. While acknowledging the limitations of the contextual surveys, the Development Panel also noted the importance of seeking ways to understand the answers to such comparative questions to situate reported results. What do students and teachers think it means to do mathematics, learn mathematics, and teach mathematics? How do teachers think about the extent to which there are opportunities to engage in authentic mathematical activity through meaningful contexts that draw out and build on students' funds of knowledge? How are teachers thinking about the mathematical strengths and mathematical identities that students bring to learning? Similarly, it may be valuable for administrator questions on the school contextual variables survey to parallel questions on the teacher survey, given the roles of administrators as instructional leaders (Cobb, Jackson, Henrick, Smith, & MIST Team, 2018; NCTM, 2014). To what extent do administrators and teachers share views on what counts as mathematical activity, including what it means to do mathematics and what it means to teach and learn mathematics, as well as questions addressing the development of mathematical identity and agency? What are the implications of teachers and administrators having different perspectives on these questions for NAEP score reporting?

One subcategory of resources for learning and instruction not specifically addressed in the NAEP Visioning Panel Guidelines document, but already present in NAEP surveys, is family perspectives on mathematics teaching and learning. Given the important influence of families on mathematics identity and agency, including the beliefs that families hold about what it means to do and learn mathematics and who has the capacity to succeed in mathematics, the panel suggested exploring how students characterize how they and their families interact around what it means to do and learn mathematics (e.g., Aguirre, Mayfield-Ingram, & Martin, 2013; Civil, 2007; Civil & Bernier, 2006; Martin, 2006).

Reflecting New Survey Emphases in New Categories

To first build an understanding of the context surveys, the Development Panel looked at the existing questions across grade levels and respondent groups (student, teacher, administrator) to determine where student, teacher, and school questions were the same and where there were differences. The next step was to cluster questions in ways that allowed an examination of the extent to which the collection of questions addressed issues of importance to the question of opportunity to learn. Finally, the third step was to look carefully across the student, teacher, and school questions to review consistency and appropriateness in how questions attended to opportunity to learn from student, teacher, and school perspectives.

As a result of this initial exploration of the mathematics-specific student surveys across grades 4, 8, and 12, the Development Panel organized existing questions into seven groups:

- Mathematics Content
- Student Engagement and Identity
- Views of Mathematics Teaching and Learning
- Features of Classroom Instruction
- Use of Technology
- Engagement in Mathematics Outside of School
- Perspectives on Family Beliefs About Mathematics Teaching and Learning

For the mathematics-specific teacher surveys for grades 4 and 8, the Development Panel identified six categories of items:

- Mathematics Content
- Views of Mathematics Teaching and Learning
- Features of Classroom Instruction Including Mathematics Teacher Learning and Support
- Use of Technology
- Student Engagement in Mathematics Outside of School
- Perspectives on Family Beliefs About Mathematics Teaching and Learning

For the mathematics-specific sections of the school surveys for grades 4, 8, and 12 that were completed by a school administrator, the Development Panel organized questions according to the following categories:

- School Mathematics Program
- Views of Mathematics Teaching and Learning
- Features of Classroom Instruction Including Mathematics Teacher Learning and Support
- Use of Technology
- Student Engagement in Mathematics Outside of School
- Perspectives on Family Beliefs About Mathematics Teaching and Learning

As shown in Illustration F.1 on the following page, all of these categories align with the Development Panel's focus on opportunities to learn and are consistent with the four issues identified in the 2017 NAEP Mathematics Survey Questionnaires. Also, as discussed in Chapter 5 of the framework, it is important to note that the panel recommended that decisions about contextual variables address the four issues in the following priority ordering (from highest priority to lowest priority):

- *Teacher preparation*, including content knowledge and subject-specific training, professional development, and non-cognitive teacher factors;
- *Student factors*, including mathematics activities outside of school, self-related beliefs, interest and motivation, grit for mathematics, and desire for learning for mathematics;
- *Resources for learning and instruction*, including people resources, product resources, and time resources; and
- *Organization of instruction*, including curriculum content, instructional strategies, use of technology in instruction, and use of formative assessment.

Illustration F.1. Issues Addressed by Surveys and Relationships to Survey Categories

	Resources for Learning and Instruction	Organization of Instruction	Teacher Preparation	Student Factors
Mathematics curriculum content	X	X	X	
Views of mathematics teaching and learning	X	X	X	X
Features of classroom instruction	X	X	X	X
Use of technology	X	X	X	
Student engagement and identity	X	X	X	X
Student engagement in mathematics outside of school				X
Family beliefs about mathematics teaching and learning				X

Attention to opportunity to learn requires foregrounding constructs such as student engagement and identity in the particular context of a mathematics classroom. In classrooms where there is limited student engagement in mathematical practices, including the NAEP Mathematical Practices, students have few opportunities to share and discuss their mathematical thinking, student perspectives are not likely to be valued and leveraged as important mathematical contributions, and the result is that students are less engaged and less able to realize their full capacity to do mathematics. It is also the case that students can shift in their levels of engagement and sense of mathematical identity as they move from one classroom to another, depending on the norms that are in place that shape what it means to teach and learn mathematics, whose voices are heard, and whose mathematical thinking matters.

Asking similar survey questions across the issues is important, as many aspects of the issues identified cut across categories and subcategories. For instance, to what extent do schools have the kinds of resources that allow teachers to engage students in mathematical activity that creates the opportunity for meaningful discussion and debate? To what extent is instruction organized in ways that allow students to meaningfully interact with classmates in small groups as they pursue solutions to tasks in ways that make sense to them? To what extent are teachers well prepared to plan and facilitate these kinds of instructional approaches, especially since these approaches often require a deeper understanding of mathematics and how students are likely to think about that mathematics? And to what extent do school administrators value and support this kind of mathematics instruction? Asking similar questions across the categories of issues may help uncover pieces that are in place, and those that are not, in ways relevant to reporting of NAEP scoring information.

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